Chapter 6

Fluids

Chapter 6

Among the most difficult graphical objects to model and animate are those that are not defined by a static, rigid, topological simple structure. Many of these complex forms are found in nature. They present especially difficult challenges for those intent on controlling their motion. In some case, special models of a specific phenomenon will suffice. We begin the chapter presenting special models for water, clouds, and fire that approximate their behavior under certain conditions. These models identify salient characteristics of the phenomena and attempt to model those characteristics explicitly.

Chapter 6

Superficial models v. Deep models

OR

Directly model visible properties	Model underlying processes that produce the visible properties
Water waves	Computational Fluid Dynamics
Wrinkles in skin and cloth	Cloth weave
Hair	Physical properties of a strand of hair
Clouds	Computational Fluid Dynamics

Chapter 6

Superficial Models for Water

Main problem with water
Changes shape
Changes topology

Still waters
Small amplitude waves
The anatomy of waves
ocean waves
running downhill

Chapter 6

Still Waters and Small-Amplitude Waves

The simplest way to model water is merely to assign the color blue to anything below a given height. If the y-axis is “up”, then color any pixel blue in which the world space coordinate of the corresponding visible surface has a y-value less than some given constant. This creates the illusion of still water at a consistent “sea level.” It is sufficient for placid lakes and puddles of standing water. Equivalently, a flat blue plane perpendicular to the y-axis and at the height of the water can be used to represent the water’s surface. These models, of course, do not produce any animation of the water.

Chapter 6

Small-Amplitude Waves

Normal vector perturbation (essentially the approach employed in bump mapping) can be used to simulate the appearance of small amplitude waves on an otherwise still body of water. To perturb the normal, one or more simple sinusoidal functions are used to modify the direction of the surface's normal vector. The functions are parameterized in terms of a single variable, usually relating to distance from the source point. It is not necessarily the case that the wave starts with zero amplitude at the source. When standing waves in a large body of water are modeled, each function usually has a constant amplitude.

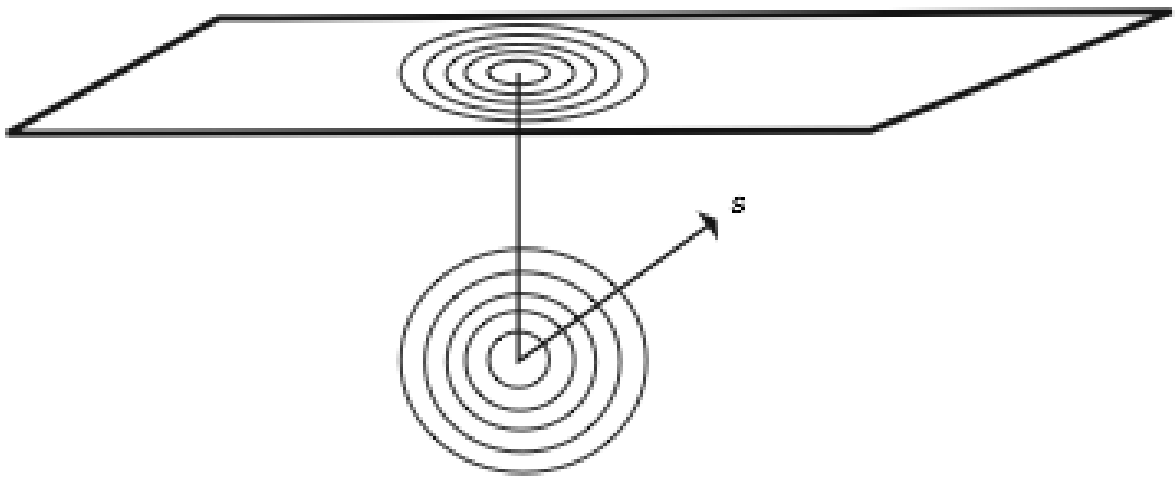
Chapter 6

Small-Amplitude Waves

The wave crests can be linear, in which case all the waves generated by a particular function travel in a uniform direction, or the wave crests can radiate from a single user-specified or randomly generated source point. Linear wave crests tend to form self-replicating patterns when viewed from a distance. For a different effect, radially symmetrical functions that help break up these global patterns can be used. Radial functions also simulate the effect of a thrown pebble or raindrop hitting the water.

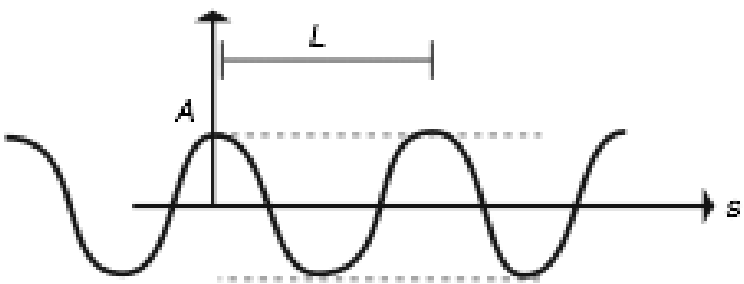
Chapter 6

Simple Wave Model - Sinusoidal



Distance-amplitude

$$h_s(s) = A\cos(\frac{s2\pi}{L})$$



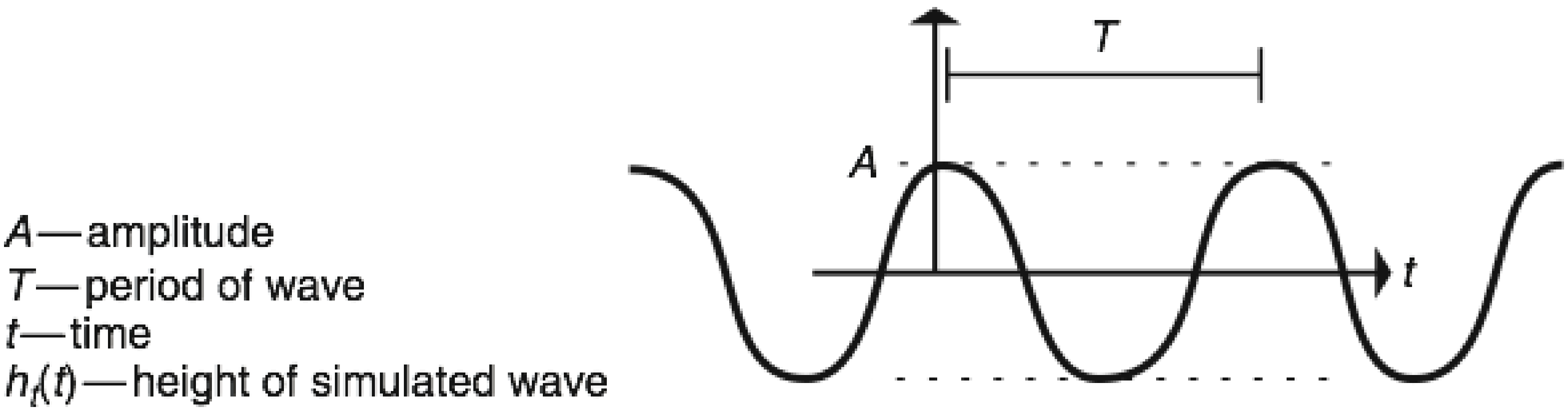
A—amplitude
L—wavelength
s—radial distance from source point
$h_s(s)$—height of simulated wave

Chapter 6

Simple Wave

The time-varying height for a point at which the wave begins at time zero is a function of the amplitude and wavelength of the wave.

$$h_t(t) = A\cos\left(\frac{2\pi t}{T}\right)$$

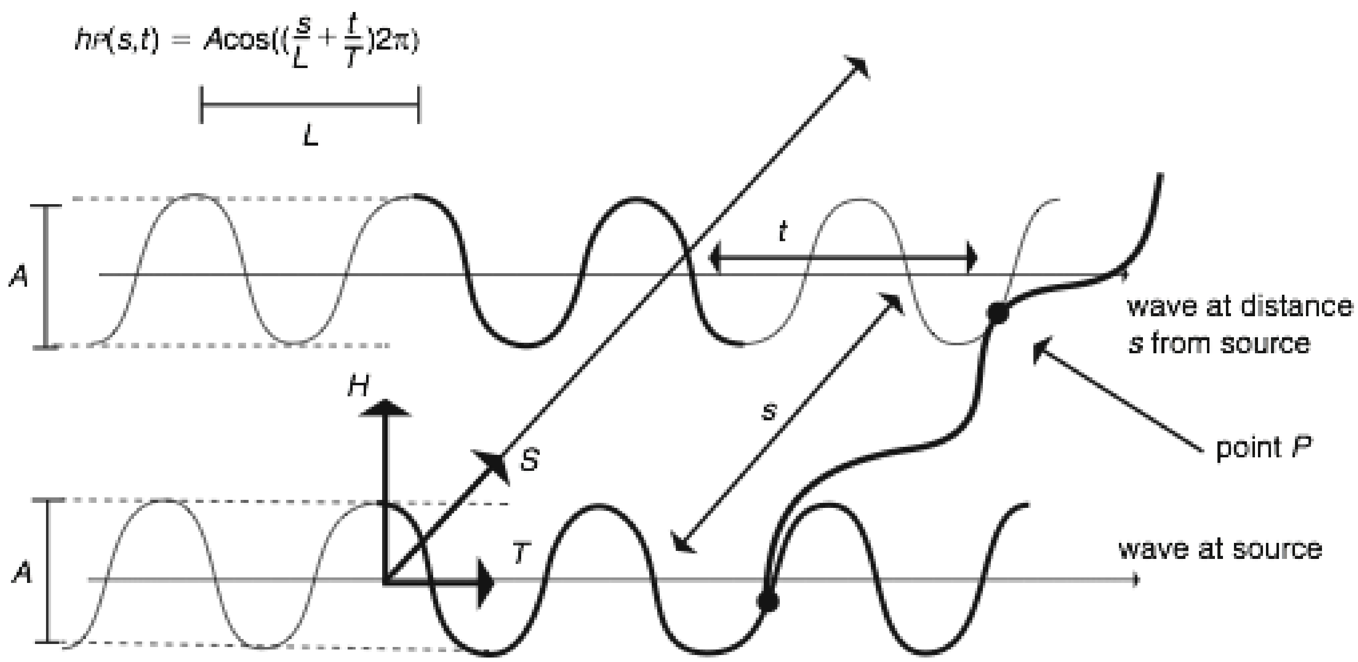


Chapter 6

Combining the two, the height of a point at some distance d from the start of the wave can be computed. This is a two-dimensional function relative to a point at which the function is zero at time zero. This function can be rotated and translated so that it is positioned and oriented appropriately in world space. Once the height function for a given point is defined, the normal to the point at any instance in time can be determined by computing the tangent vector and forming the vector perpendicular to it. These vectors should then be oriented in world space, so the plane they define contains the direction that the wave is traveling.

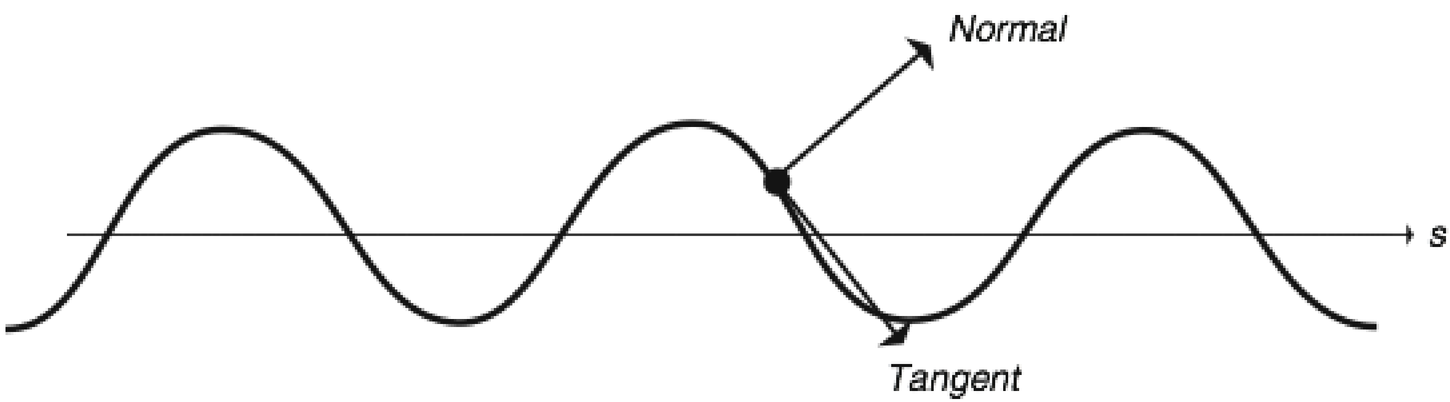
Chapter 6

Simple Wave



Chapter 6

Simple Wave



$$h_p(s,t) = A\cos\left(\left(\frac{s}{L} + \frac{t}{T}\right)2\pi\right)$$

$$\frac{d}{d_s}h_p(s,t) = -\left(A\frac{2\pi}{L}\sin\left(\left(\frac{s}{L} + \frac{t}{T}\right)2\pi\right)\right)$$

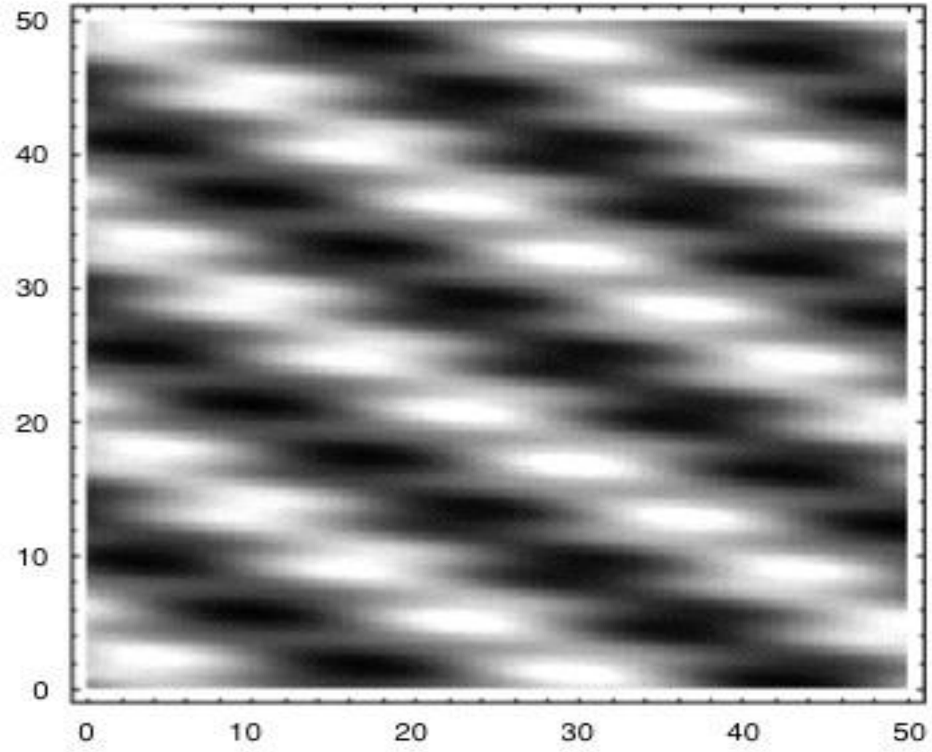
$$Tangent = \left(1, \frac{d}{ds}h_p(s,t), 0\right)$$

$$Normal = \left(-\left(\frac{d}{ds}h_p(s,t)\right)1, 0\right)$$

Chapter 6

Sum of Sinusoidals

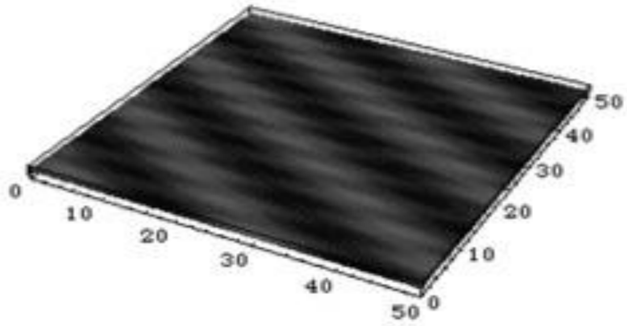
Superimposing multiple sinusoidal functions of different amplitude and with various source points can generate interesting patterns of overlapping ripples.



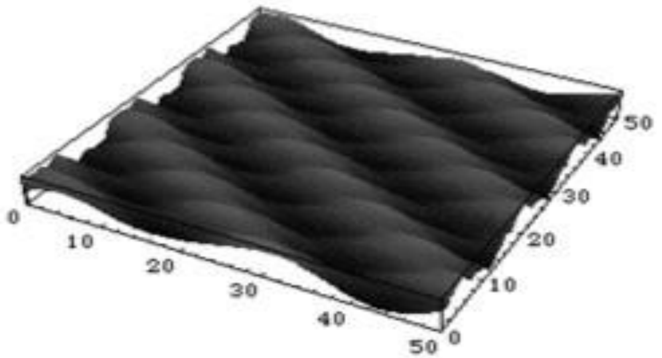
Chapter 6

Sum of Sinusoidals

Calculating the normals without changing the surface creates the illusion of waves on the surface of the water. However, whenever the water meets a protruding surface, such as a rock, the lack of surface displacement will be evident.



Normal vector perturbation



Height field

Chapter 6

The same approach used to calculate wave normals can be used to modify the height of the surface. A mesh of points can be used to model the surface of the water and the heights of the individual points can be controlled by the overlapping sinusoidal functions. Either a faceted surface with smooth shading can be used or the points can be the control points of a higher-order surface such as a B-spline surface. The points must be sufficiently dense to sample the height function accurately enough for rendering.

Similarly, displacement mapping can be used to achieve the effect of waves.

Chapter 6

Ocean Waves

$$F(s,t) = A\cos\left(\frac{2\pi(s - Ct)}{L}\right)$$

s – distance from source
t – time
A – maximum amplitude
C – propagation speed
L – wavelength
T – period of wave

Relationship between wavelength, period, and speed:

$C=L/T$

Chapter 6

Ocean Waves

The motion of the wave is different from the motion of the water. The wave travels linearly across the surface of the water, while a particle of water moves in nearly a circular orbit. While riding the crest of the wave, the particle will move in the direction of the wave. As the wave passes and the particle drops into the trough of the waves, it will travel in the reverse direction. The steepness, S, of the wave is represented by the term H/L where H is defined as half of the amplitude.

Waves with a small steepness value have basically a sinusoidal shape. As the steepness value increases, the shape of the wave gradually changes into a sharply crested peak with flatter troughs.

Chapter 6

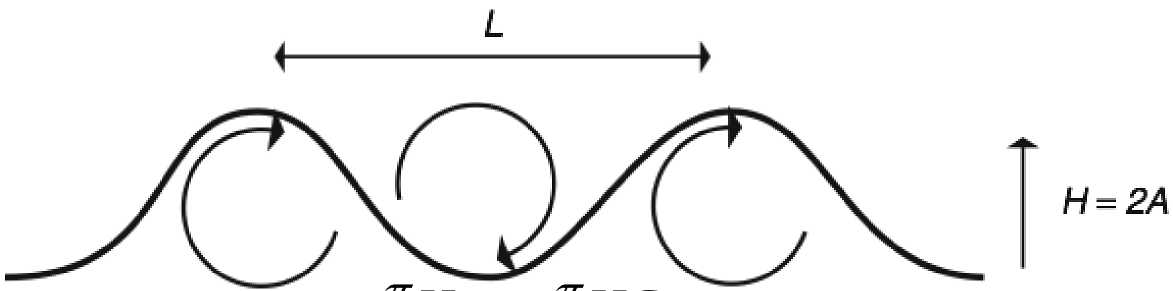
Idealized Wave

In an idealized wave, there is no transport of water. The particle of water completes one orbit in the time it takes for one complete cycle of the wave to pass. The average orbital speed of a particle of water is given by the circumference of the orbit, πH, divided by the time it takes to complete the orbit, T.

Chapter 6

Movement of a particle

In idealized wave, no transport of water



$$Q_{average} = \frac{\pi H}{T} = \frac{\pi HC}{L} = \pi SC$$

Q – average orbital speed
S – steepness of the wave
T – time to complete orbit
H – twice the amplitude

Chapter 6

Breaking Wave

If the orbital speed, Q, of the water at the crest exceeds the speed of the wave, C, then the water will spill over the wave, resulting in a breaking wave. Because the average speed, Q, increases as the steepness, S, of the wave increases, this limits the steepness of a non-breaking wave, The observed steepness of ocean waves is between 0.5 and 1.0.

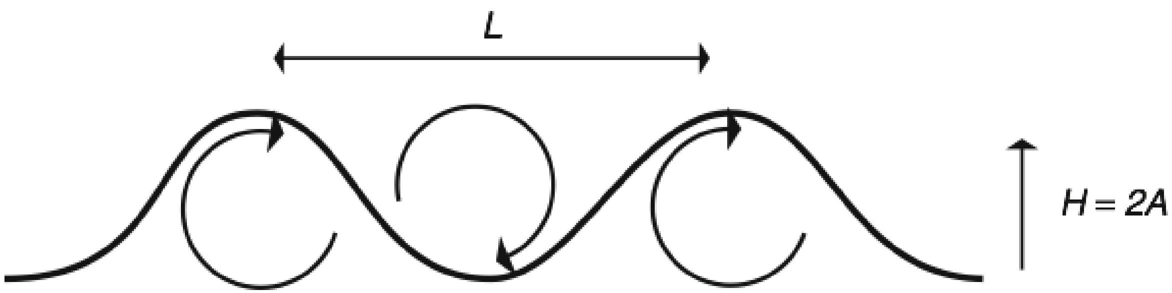
Chapter 6

Breaking waves

If Q exceeds C => breaking wave

If non-breaking wave, steepness is limited

Observed steepness between 0.5 and 1.0



$$Q_{average} = \frac{\pi H}{T} = \frac{\pi HC}{L} = \pi SC$$

Chapter 6

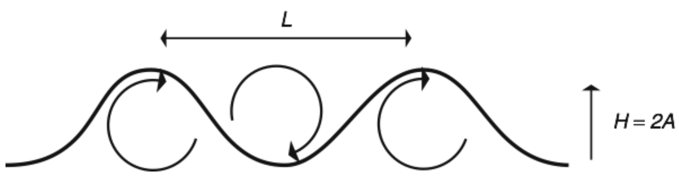
Breaking Waves

A common simplification of the full CFD simulation of ocean waves is called the Airy model, and it relates the depth of the water, d, the propagation speed, C, and the wavelength of the wave, L.

Chapter 6

Airy model of waves

Relates depth of water, propagation speed and wavelength



$$C = \sqrt{\frac{g}{\kappa}\tanh(\kappa d)} = \sqrt{\frac{gL}{\kappa\pi}\tanh(\frac{2\pi d}{L})}$$

$$L = CT$$

$$\kappa = \frac{2\pi}{L}$$

g – gravity:
9.81 m/s²
(at sea level)

As depth increases, C approaches $\sqrt{\frac{gL}{2\pi}}$
(*tanh(κd)* tends toward one)

As depth decreases, C approaches $\sqrt{gd}$
(*tanh(κd)* approaches *κd*)

Chapter 6

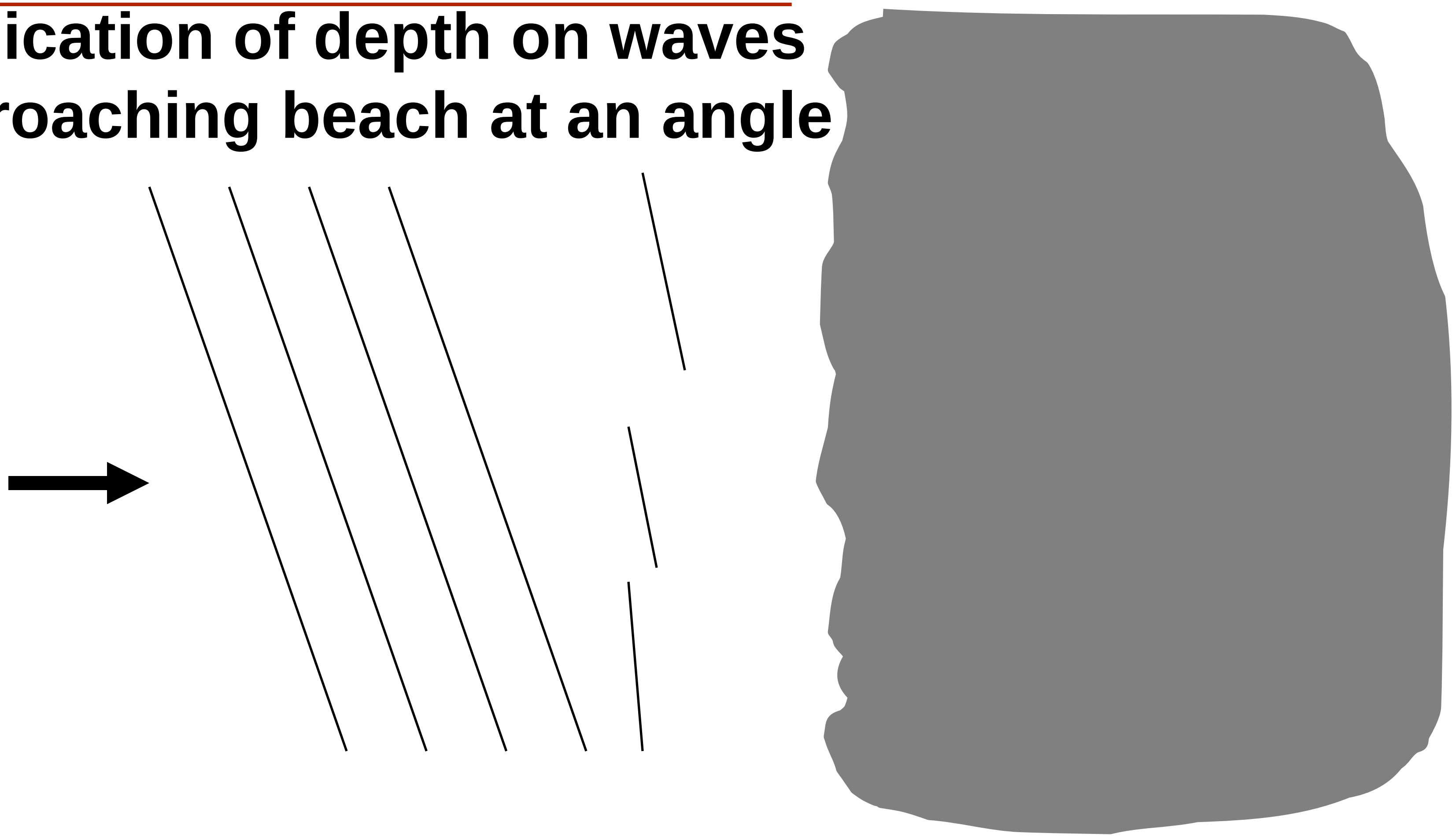
Airy model

As a wave approaches the shoreline at an angle, the part of the wave that approaches first will slow down as it encounters a shallower area. The wave will progressively slow down along its length as more of it encounters the shallow area. This will tend to straighten out the wave and is called *wave refraction*.

Interestingly, even though speed and wavelength of the wave are reduced as the wave enters shallow water, the period of the wave remains the same and the amplitude remains the same or increases. As a result, the orbital speed of the water remains the same. Because orbital speed remains the same as the speed of the wave decreases, waves tend to break as they approach the shoreline because the speed of the water exceeds the speed of the wave.

Chapter 6

Implication of depth on waves approaching beach at an angle



Wave tends to straighten out – closer sections slow down

Chapter 6

Wave in shallow water

C and L are reduced

T remains the same

A (H) remain the same or increase

Q remains the same

Waves break

Chapter 6

Example: Happy Feet

Once the wave breaks, we need - besides the geometry that is required for creating a realistic looking wave - a particle spray at the front of the wave.

Chapter 6

Model to Transport of Water

One of the assumption used to model ocean waves is that there is no transport of water. However, in many situations, such as a stream of water running downhill, it is useful to model how water travels from one location to another. In situations in which the water can be considered a height field and the motion assumed to be uniform through a vertical column of water, the vertical component of the velocity can be ignored. In such cases, differential equations can be used to simulate a wider range of convincing motion. The Navier-Stokes equations (which describe flow through a volume) can be simplified to model the flow.

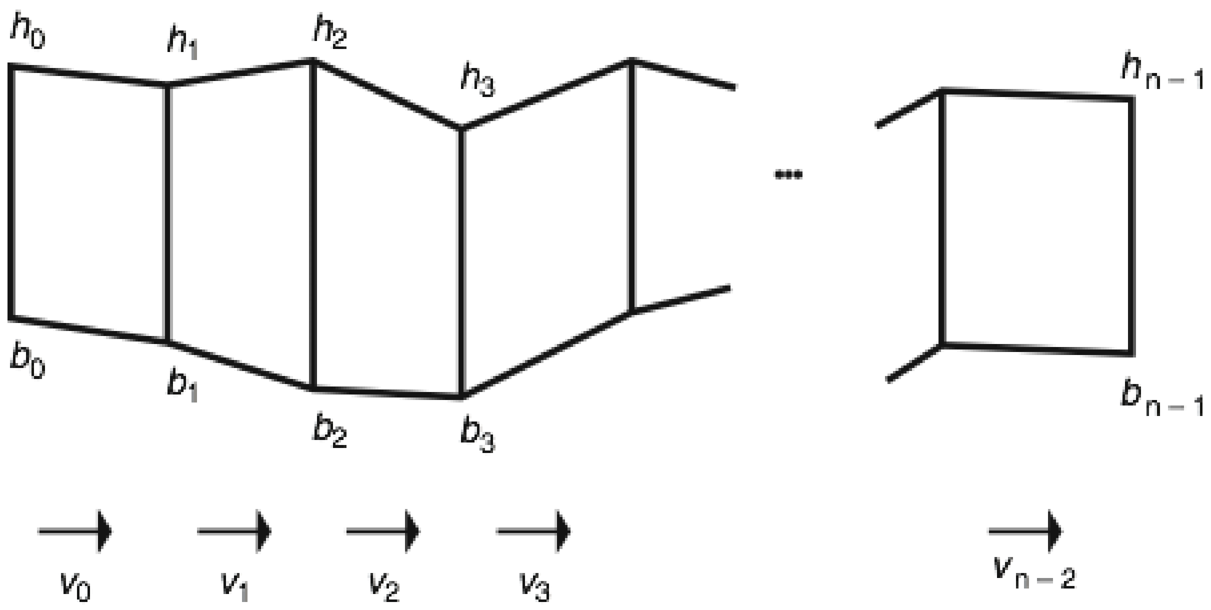
Chapter 6

Model to Transport of Water

To develop the equations in two dimensions, the user parameterizes functions that are in terms of distance x. Let $z=h(x)$ be the height of the water and $z=b(x)$ the height of the ground at location x. The height of the water is $d(x)=h(x)-b(x)$. We then assume that motion is uniform through a vertical column of water and that $v(x)$ is the velocity of a vertical column of water.

Chapter 6

Model for Transport of Water

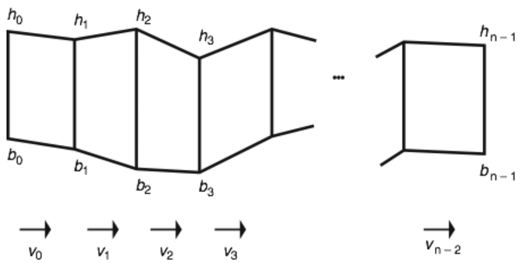


h – water surface
b – ground
v – water velocity

Chapter 6

Model for Transport of Water

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + g\frac{\partial h}{\partial x} = 0$$

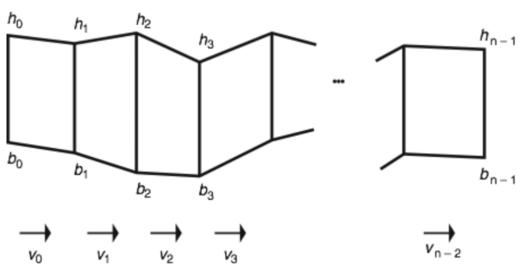


Here, g is the gravitation acceleration. The equation considers the change in velocity of the water and relates its acceleration, the difference in adjacent velocities, and the acceleration due to gravity when adjacent columns of water are at different heights.

Chapter 6

Model for Transport of Water

$$\frac{\partial d}{\partial t} + \frac{\partial (vd)}{\partial x} = 0$$



$$d = h(x) - b(x)$$

This equation considers the transport of water by relating the temporal change in the height of the vertical column of the water with the spatial change in the amount of water moving.

Chapter 6

Model to Transport of Water

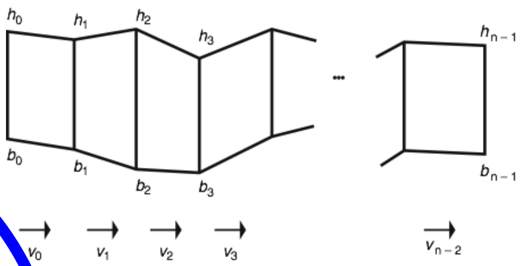
These equations can be further simplified if the assumptions of small fluid velocity and varying depth are used. The former assumption eliminates the second term of the equation on the second last slide, while the latter assumption implies that the term d can be removed from inside the derivative in the previous equation. These simplifications result in the following:

Chapter 6

Model for Transport of Water

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + g\frac{\partial h}{\partial x} = 0$$

$$\frac{\partial d}{\partial t} + \frac{\partial (vd)}{\partial x} = 0$$



$$\frac{\partial d}{\partial t} + d\frac{\partial (v)}{\partial x} = 0$$

Differentiate w.r.t. x

Differentiate w.r.t. t

$$\frac{\partial^2 h}{\partial t^2} = gd\frac{\partial^2 h}{\partial x^2}$$

Small fluid velocity

Slowly varying depth

Use finite differences to model, i.e. discretize differentials - see book

Chapter 6

Models for Clouds

Modeling clouds is a very difficult task because of their complex, amorphous, space-filling structure and because even an untrained eye can easily judge the realism of a cloud model. The ubiquitous nature of clouds makes them an important modeling and animation task. This section describes their important visual and physical characteristics, important rendering issues, and several approaches for cloud modeling.

Chapter 6

Models for Clouds

- Basic cloud types
- Physics of clouds
- Visual characteristics, rendering issues
- Early approaches
- Volumetric cloud modeling

Chapter 6

Basic Cloud Types

Example forces of formation:

- Convection
- Convergence lifting along frontal boundaries
- Lifting due to mountains

Height – water droplet v. ice crystal composition

The shape of the cloud varies based on processes that force the air to rise or bubble up and the height at which the cloud forms

Chapter 6

Clouds formed above 20,000 feet (**cirrus**) are wispy and white in appearance and composed primarily of ice crystals. Clouds formed between 6,500 feet and 23,000 feet (**altocumulus**) are primarily composed of water droplets; they are small and puffy and they collect into groups, sometimes forming waves. Clouds formed below 6,500 feet (**stratus**, **stratocumulus**) are again composed primarily of water droplets; they extend over a large area and have a layered or belled appearance. The most characteristic cloud type is the puffy **cumulus**. Cumulus clouds are normally formed by convection or frontal lifting and can vary from having little vertical height to forming huge vertical towers (**cumulonimbus**) created by strong convection.

Chapter 6

Visual characteristics

3D
Amorphous
Turbulent
Complex shading
- Semi-transparent
- Self-shadowing
- Reflective (albedo)

Chapter 6

Early Approach - Gardner

Early flight simulator research
Static model for the most part
Sum of overlapping semi-transparent hollow ellipsoids
Taper transparency from edges to center

See his paper from SIGGRAPH 1985

Chapter 6

Early Approach - Gardner

Chapter 6

Other approaches

Particle systems
implicit functions
Volumetric representations

Chapter 6

Volumetric Cloud Modeling

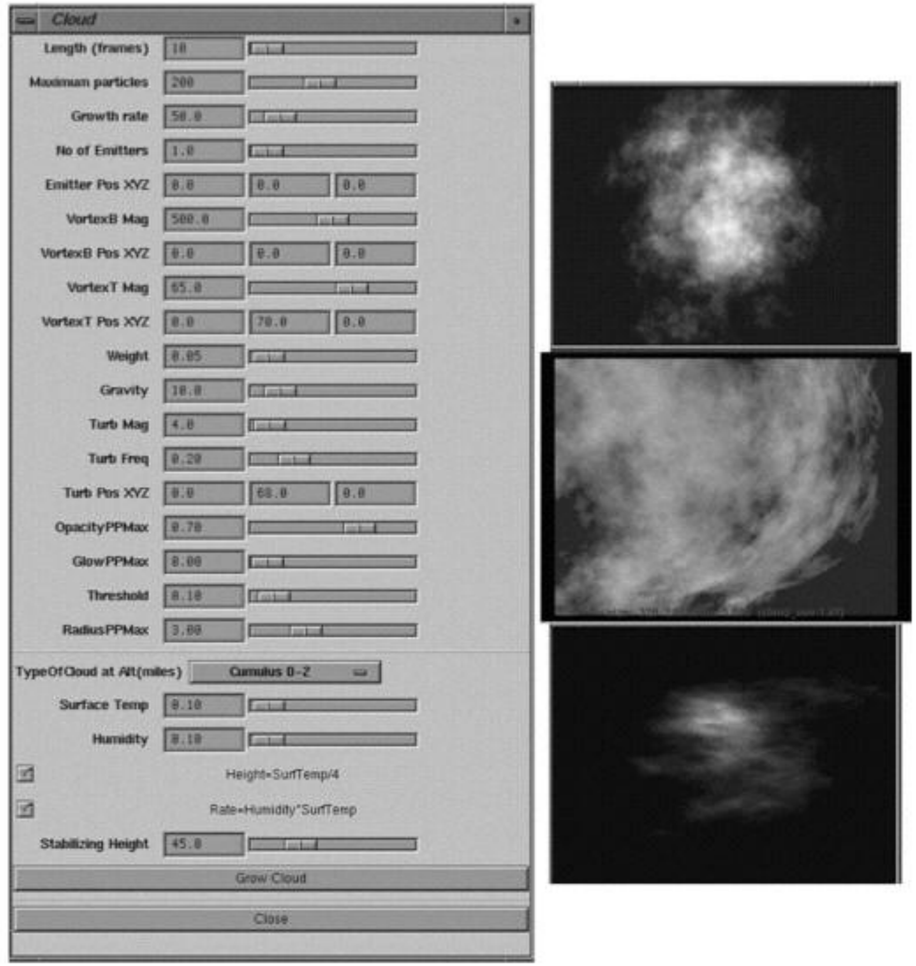
Although surface-based techniques can produce realistic images of clouds viewed from a distance, these cloud models are hollow and do not allow the user to seamlessly enter, travel through, and inspect their interior. Volumetric density-based models must be used to capture the three-dimensional structure of a cloud.

Particle systems are commonly used to simulate the volumetric gases, such as smoke, with convincing results and provide easy animation control. The difficulty with using particle systems for cloud modeling is the massive number of particles that is necessary to simulate realistic clouds.

Chapter 6

Animating Volumetric Procedural Clouds

An example of cloud dynamics GUI and example images created in Maya



(a) GUI used to control cloud formation (b) Example clouds

Chapter 6

Volumetric Clouds (David Ebert)

Ebert’s cloud modeling and animation approach uses procedural abstraction of detail to allow the designer to control and animate objects at high level. Its inherent procedural nature provides flexibility, data amplification, abstraction of detail, and ease of parametric control.

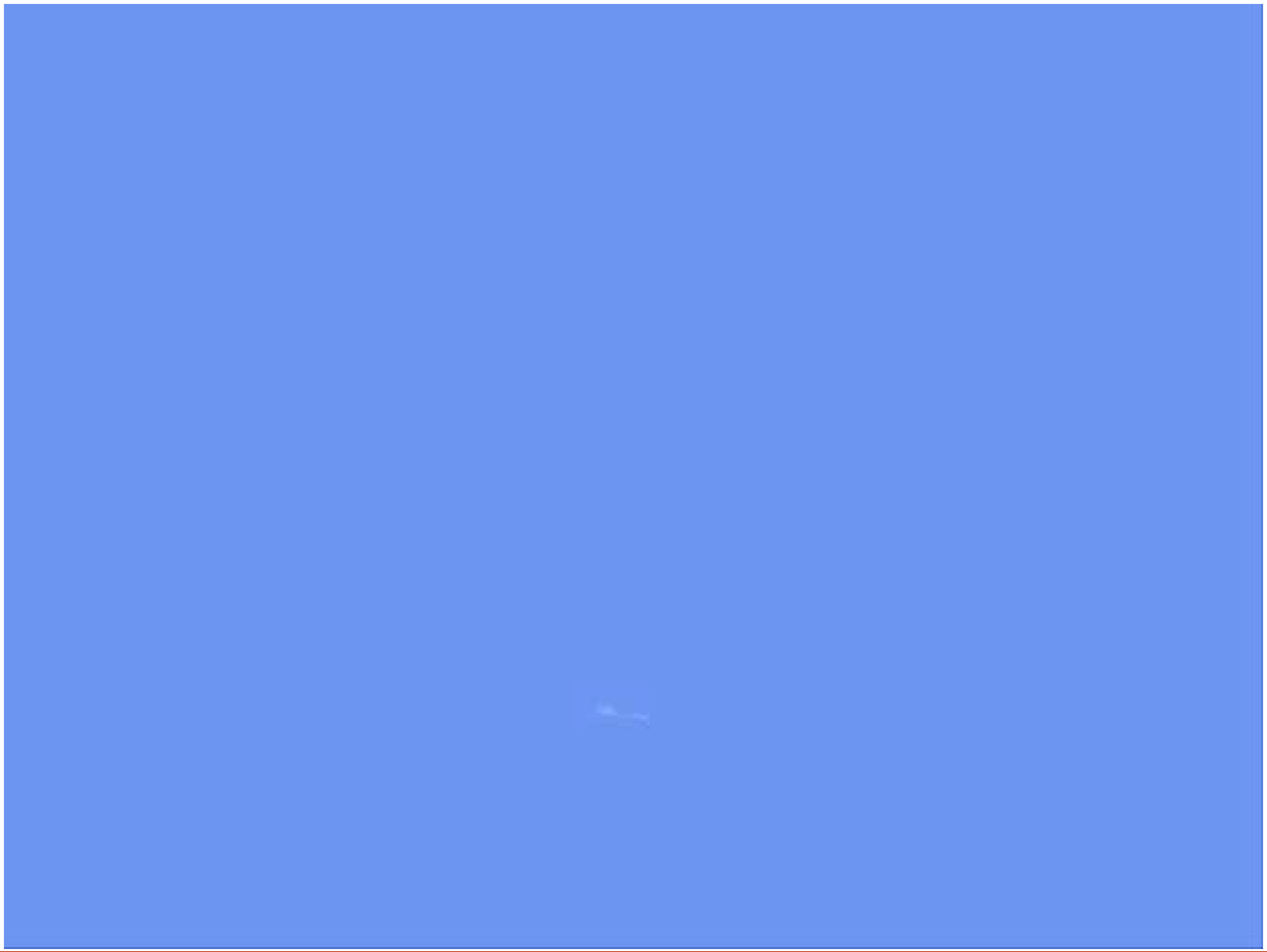
Sample clouds (cirrus and cirrostratus)

Chapter 6

Volumetric Clouds (David Ebert)

Chapter 6

Volumetric Clouds: results (David Ebert)



Chapter 6

Models for Fire

Procedural 2D
Particle system
Other approaches

Chapter 6

Models of Fire

Fire is a particularly difficult and computationally intensive process to model. It has all the complexities of smoke and clouds and the added complexity of very active internal processes that produce light and motion and create rapidly varying display attributes. Fire exhibits temporally and spatially transient features at a variety of granularities. The underlying process is that of combustion – a rapid chemical process that releases heat and light accompanied by flame. A common example is that of a wood fire in which the hydrocarbon atoms of the wood fuel join with the oxygen atoms to form water vapor, carbon monoxide, and carbon dioxide. As a consequence of the heated gases rising quickly, air is sucked into the combustion area creating turbulence.

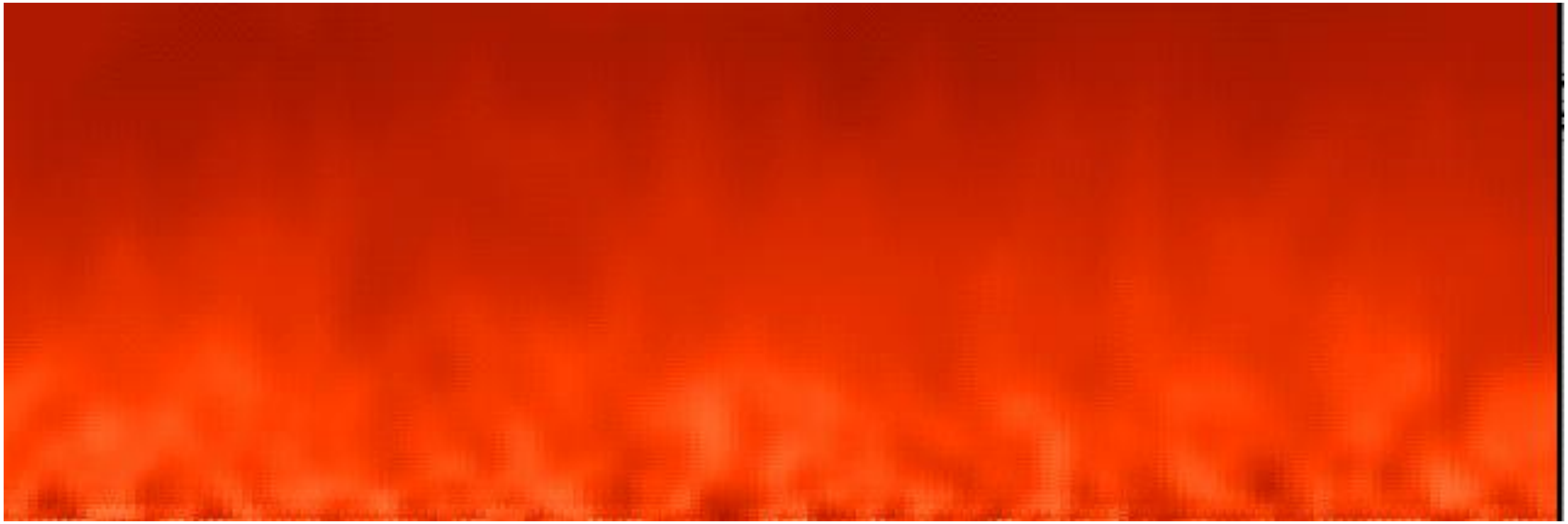
Chapter 6

Procedurally Generated Image

A particularly simple way to generate the appearance of fire is to procedurally generate a two-dimensional image by coloring pixels suggestive of flames and smoke. The procedure iterates through pixels of an image buffer and sets the palette indices based on the index of the surrounding pixels. Modifying the pixels top to bottom allows the imagery to progress up the buffer. By using multiple buffers and the alpha channel a limited three-dimensional effect can be achieved. In the following image, a color palette filled with hues from red to yellow is used to hold RGB values. The bottom row of the image buffer is randomly initialized with color indices.

Chapter 6

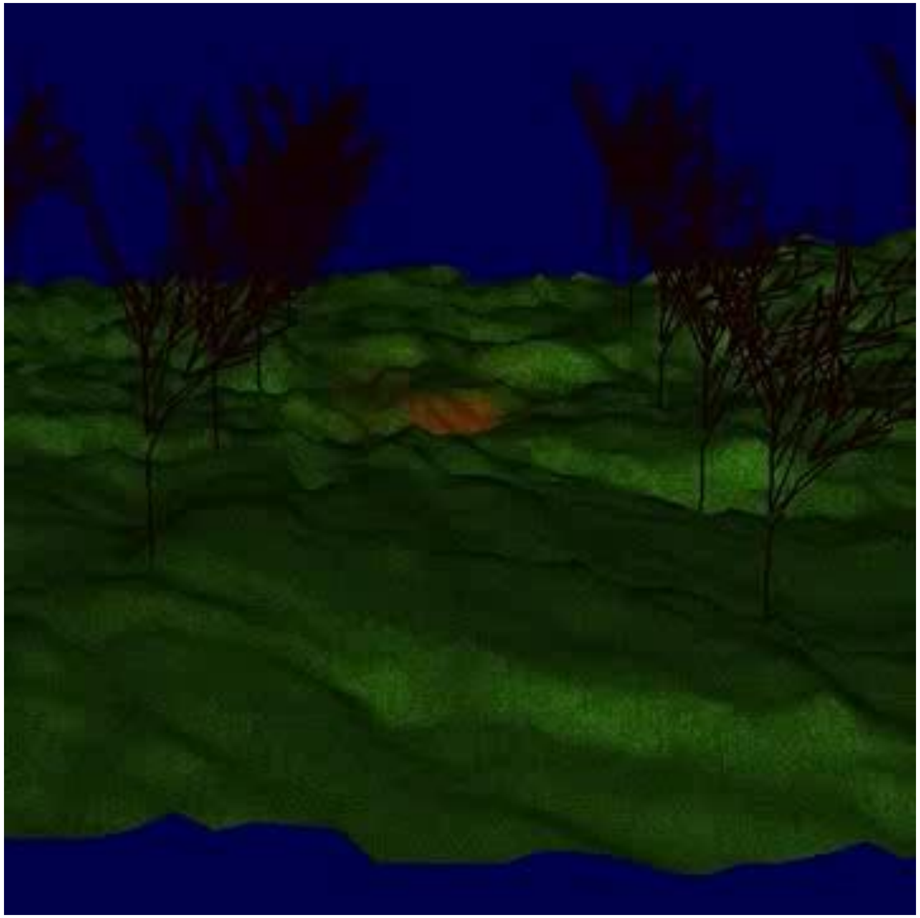
Models for Fire - 2D



Chapter 6

Models for Fire

link



Chapter 6

Particle System Approach

One of the first and most popularly viewed examples of computer-generated fire appears in the movie *Start Trek II: Wrath of Kahn*. In the sequence referred to as the *genesis effect*, an expanding wall of fire spreads out over the surface of a planet from a single point of impact. The simulation is not a completely convincing model of fire, although the sequence is effective in the movie. The model uses a two-level hierarchy of particles. The first level of particles is located at the point of impact to simulate the initial blast; the second level consists of concentric rings of particles, timed to progress from the central point outward, forming the wall of fire and explosions.

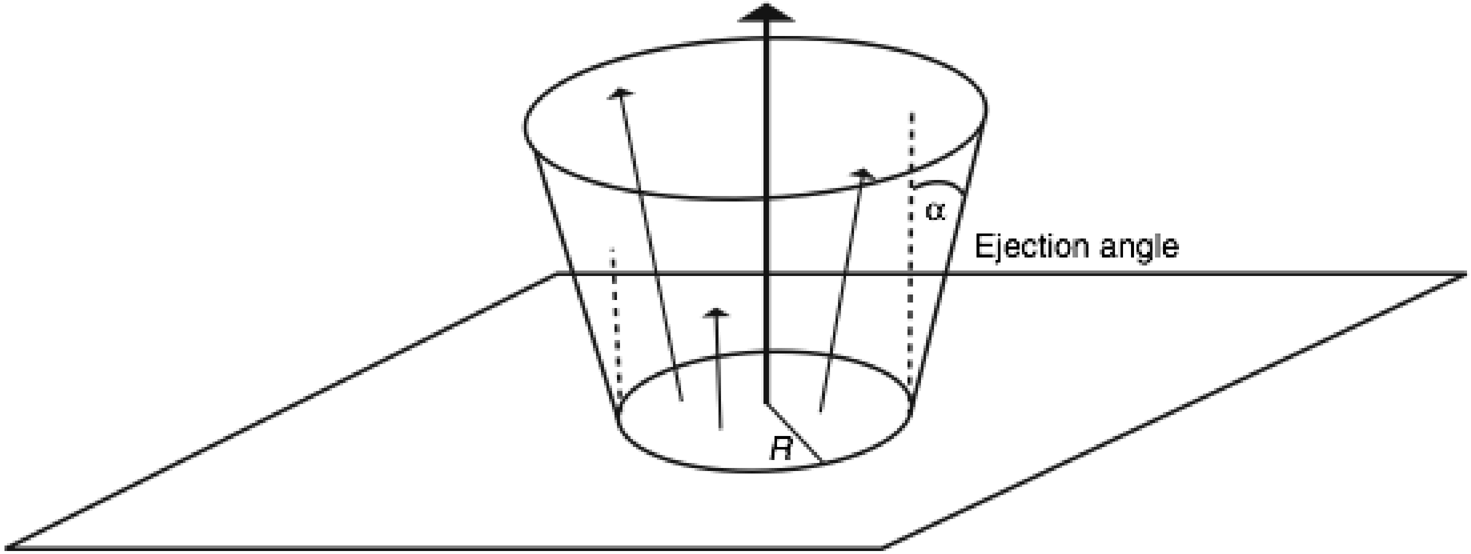
Chapter 6

Particle System Approach

Each ring of second-level hierarchy consists of a number of individual particle systems that are positioned on the ring and overlaps as to form a continuous ring. The individual particle systems are modeled to look like explosions. The particles in each one of these particle systems are oriented to fly up and away from the surface of the planet. The initial position for a particle is randomly chosen from the circular base of the particle system. The initial direction of travel for each particle is constrained to deviate less than the ejection angle away from the surface normal

Chapter 6

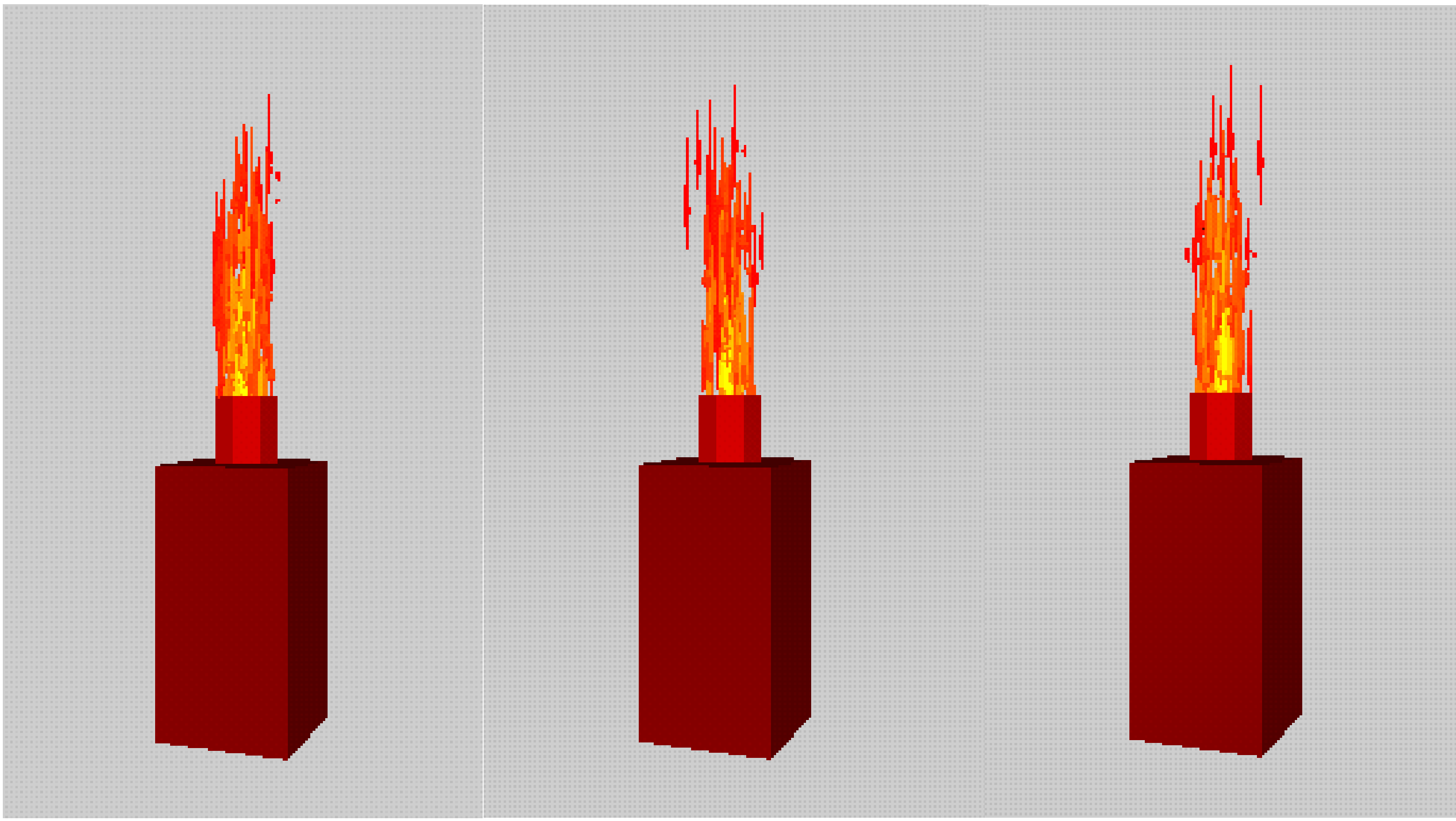
Models for Fire - particle system



Derived from Reeves' paper on particle systems

Chapter 6

Particle System Fire



Chapter 6

Star Trek: Wrath of Kan – Project Genesis

video

Chapter 6

Com-bustion examples

Chapter 6

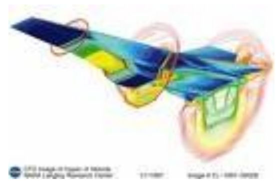
Com-bustion examples

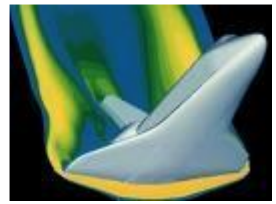
Chapter 6

Combustion examples

GPU fluid simulation - fire

Computational Fluid Dynamics (CFD)





Fluid - a substance, as a liquid or gas, that is capable of flowing and that changes its shape at a steady rate when acted upon by a force.

Compressible – changeable density

Steady state flow – motion attributes (e.g. velocity and acceleration) are constant at a point

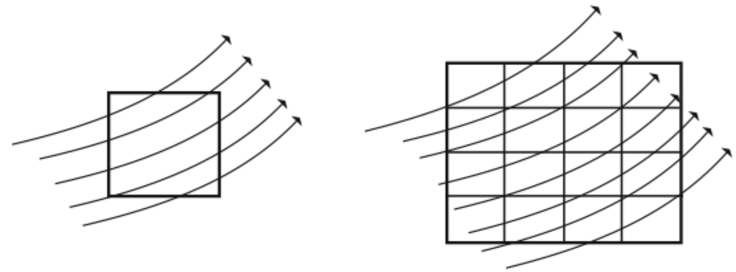
Viscous – resists flow; **Newtonian** fluid has linear stress-strain rate

Vortices – circular swirls

Chapter 6

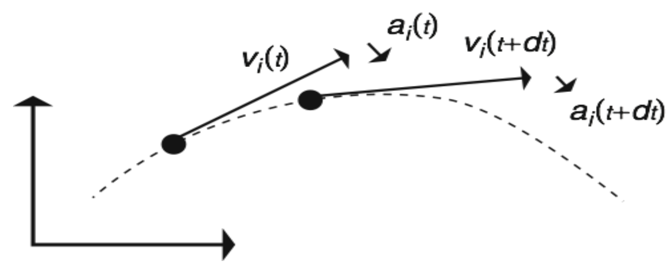
General Approaches

Grid-based

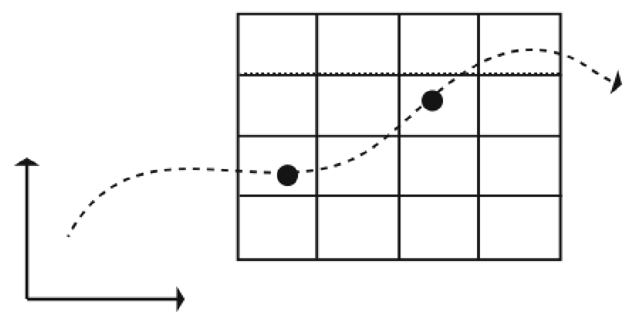


a) Gas flowing through an individual cell   b) Grid of cells

Particle-based method



Hybrid method



Chapter 6

Grid-Based Method

The grid-based method decomposes space into individual cells, and the flow of the gas into and out of each cell is calculated. In this way, the density of gas in each cell is updated from time step to time step. The density in each cell is used to determine visibility and illumination of the gas during rendering. Attributes of the gas within a cell, such as velocity, acceleration, and density, can be used to track the gas as it travels from cell to cell.

Chapter 6

Grid-Based Method

The flow out of the cell can be computed based on the cell velocity, the size of the cell, and the cell density. The flow into a cell is determined by distributing the densities out of adjacent cells. External forces, such as wind and obstacles, are used to modify the acceleration of the particles within a cell. The rendering phase uses standard volumetric graphics technique to produce an image based on the densities projected onto the image plane. Illumination of a cell from a light source through the volume must also be incorporated into a display procedure.

Chapter 6

Particle-Based Method

Particles or globs are tracked as they progress through space, often with a standard particle system approach. The particles can be rendered individually, or they can be rendered as spheres of gas with a given density. The advantage of this technique is that it is similar to rigid body dynamics and therefore the equations are relatively simple and familiar. The equations can be simplified if the rotational dynamics are ignored. In addition, There are no restrictions imposed by the simulation setup as to where the gas may travel. External forces can be easily incorporated. The disadvantage of this approach is that a large number of particles is needed.

Chapter 6

Hybrid Method

Some models of gas trace particles through a spatial grid. Particles are passed from cell to cell as they traverse the interesting space. The display attributes of individual cells are determined by the number and type of particles contained in the cell at the time of display. The particles are used to carry and distribute attributes through the grid, and then the grid is used to produce the display.

Chapter 6

CFD equations

mass is conserved
momentum is conserved
energy is conserved ←

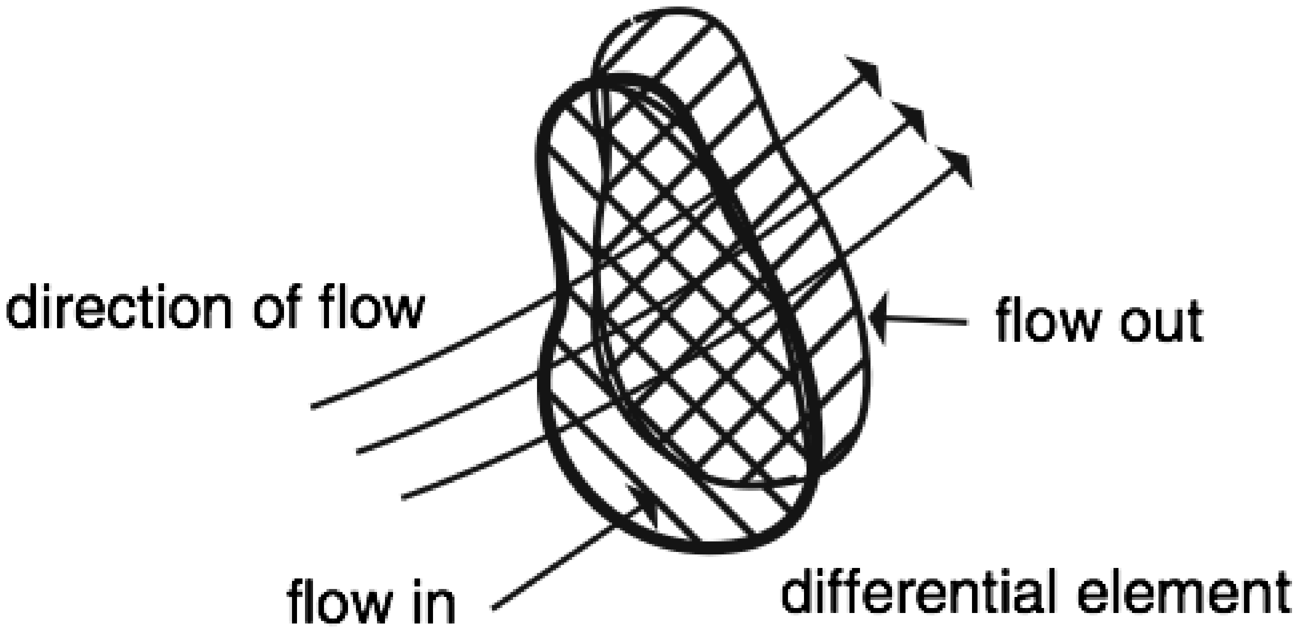
Usually not modeled in computer animation

To solve:

- discretize cells
- discrete equations
- numerically solve

Chapter 6

CFD



Differential element used in Navier-Stokes

Chapter 6

Simplifications

The Navier-Stokes equations are non-linear differential equations which can be difficult to solve. Hence, the following simplifications are often made:

- Ignore energy conservation
- Non-viscous fluid
- Incompressible flow

By ignoring energy conservation and viscosity, the Euler equations describing the conservation of mass and conversation of momentum can be readily constructed for a given field.

Chapter 6

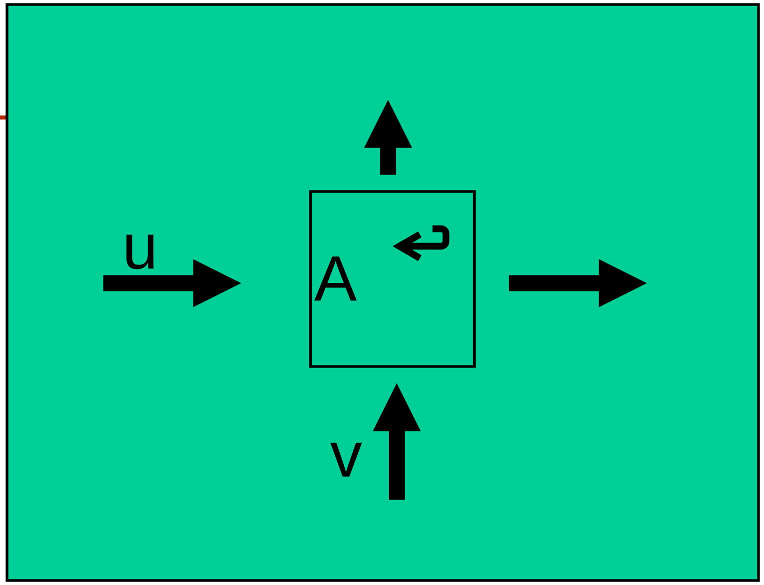
Conservation of mass (2D)

In order to keep it a little simpler, let's look at the 2D case (which then can be extended to 3D).

For now, consider only flow in the x-direction. To make the discussion a bit more concrete, assume that the fluid is flowing left to right, as x increases, so that the mass flows into a cell on the left, at x, and out the cell on the right, at $x+dx$.

In the following equations, ρ is the density, p is the pressure, and A is the area of the cell.

Conservation of mass (2D)

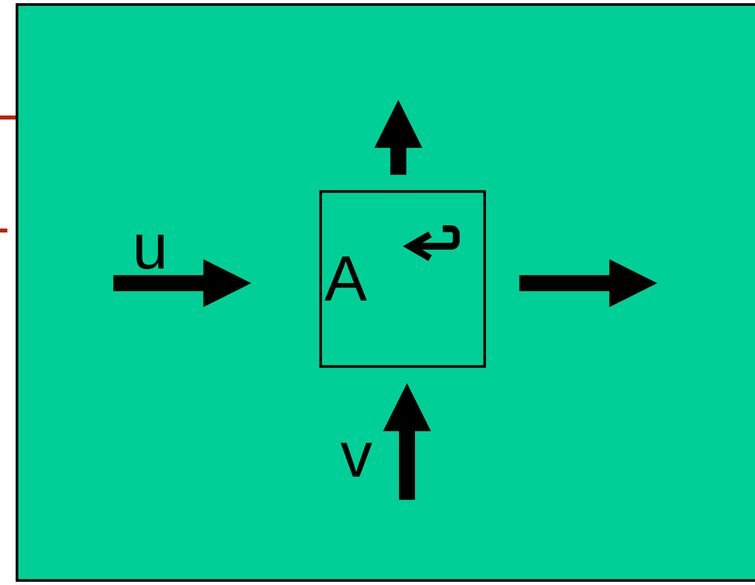


Small control volume: Δx by Δy by 1
A = Δx * Δy

Time rate of mass change in volume = rate of mass entering

Mass inside CV: $\rho \cdot \Delta x \cdot \Delta y$

Time rate of mass change in volume → $\frac{\partial(\rho \Delta x \Delta y)}{\partial t} = \Delta x \Delta y \frac{\partial(\rho)}{\partial t}$



Conservation of mass (2D)

Small control volume: Δx by Δy by 1
A = Δx * Δy

Time rate of mass change in volume = rate of mass entering

$$\frac{\partial(\rho \Delta x \Delta y)}{\partial t} = \Delta x \Delta y \frac{\partial \rho}{\partial t}$$

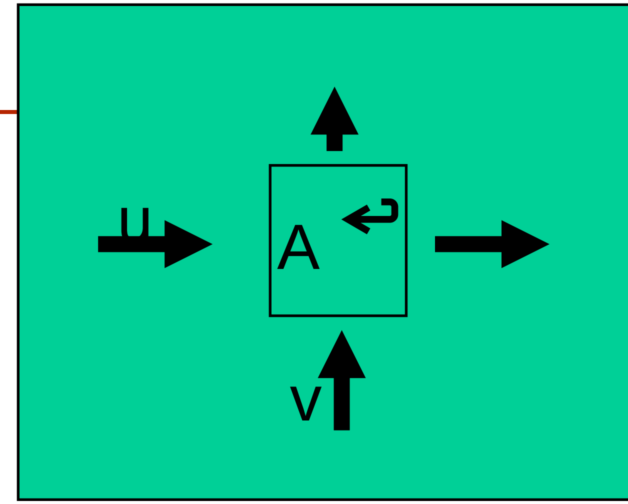
Amount of mass entering from left: $\rho \frac{dx}{dt} \Delta y = \rho u \Delta y$

Difference between left and right:

$$((\rho u)_{left} - (\rho u)_{right}) \Delta y = (\rho u - \rho u - d(\rho u)) \Delta y = -d(\rho u) \Delta y$$

Chapter 6

Conservation of mass (2D)



Time rate of mass change in volume = rate of mass entering

$$\frac{\partial(\rho)}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

$$\frac{\partial(\rho)}{\partial t} + \nabla \cdot (\rho V) = 0$$

Divergence operator

If incompressible $0 = \nabla \cdot V$

Chapter 6

Conservation of Momentum

The momentum of an object is its mass times its velocity. The momentum of a fluid is its density times the volume of the fluid times the average velocity of the fluid.

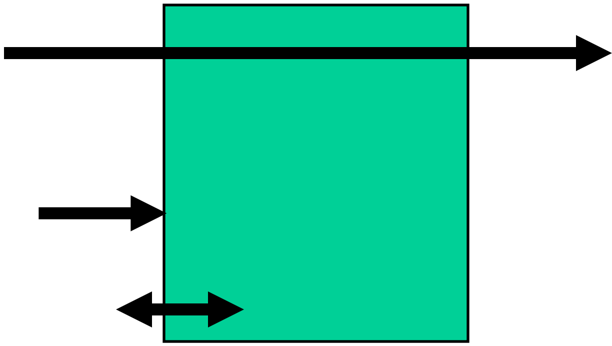
Chapter 6

Conservation of momentum

Momentum in CV changes as the result of:

Mass flowing in and out

Collisions of adjacent fluid (pressure)



Random interchange of fluid at boundary

Chapter 6

Conservation of momentum in 2D

Rate of change of u-momentum within CV $\frac{\partial(\rho u)}{dt}$

u-Momentum entering: $mV_x = (\rho u)u$

Difference in x: $\frac{\partial(\rho u^2)}{dx}$

Difference in y: $\frac{\partial(\rho uv)}{dy}$

Pressure difference in x : $\frac{\partial(p)}{dx}$

Note: equations already divided by volume $V=dxdy$

Chapter 6

Conservation of momentum in 2D

$$\frac{\partial(\rho u)}{dt}+\frac{\partial(\rho u^2)}{dx}+\frac{\partial(\rho uv)}{dy}+\frac{\partial(p)}{dx}=0$$

$$\frac{\partial(\rho v)}{dt}+\frac{\partial(\rho uv)}{dx}+\frac{\partial(\rho v^2)}{dy}+\frac{\partial(p)}{dy}=0$$

Chapter 6

Conservation of momentum (3D)

$$-\frac{dp}{dx} = \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho vu)}{\partial y} + \frac{\partial(\rho wu)}{\partial z} + \frac{\partial(\rho u)}{\partial t}$$ x direction

$$-\frac{dp}{dy} = \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho v^2)}{\partial y} + \frac{\partial(\rho wv)}{\partial z} + \frac{\partial(\rho v)}{\partial t}$$ y direction

$$-\frac{dp}{dz} = \frac{\partial(\rho uw)}{\partial x} + \frac{\partial(\rho vw)}{\partial y} + \frac{\partial(\rho w^2)}{\partial z} + \frac{\partial(\rho w)}{\partial t}$$ z direction

$$-\nabla p = \rho\left(\frac{\partial V}{\partial t} + V \cdot \nabla V\right) = \rho\dot{V} = \rho\frac{DV}{Dt}$$ Material derivative

Chapter 6

Navier-Stokes (for graphics)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V) = 0$$

$$-\nabla p = \rho\left(\frac{\partial V}{\partial t} + V \cdot \nabla V\right) = \rho \dot{V} = \rho \frac{Dv}{Dt}$$

$$0 = \nabla \cdot V$$

Chapter 6

Solving the Equations

Before solving these equations, boundary conditions must be set up to handle the cells at the limits of the domain. For example, the **Dirichlet boundary condition**, which sets solution values at the boundary cells, is commonly used. The CFD equations, set up for each cell for the grid, produce a sparse set of matrix equations. There are various ways to solve such systems. For example, LU and Conjugate Gradient solvers are often used. Efficient, accurate methods of solving symmetric and asymmetric sparse matrices, such as these, are the topic of ongoing research. Thankfully in computer animation, believability is more of a concern than accuracy, so approximate techniques often suffice.

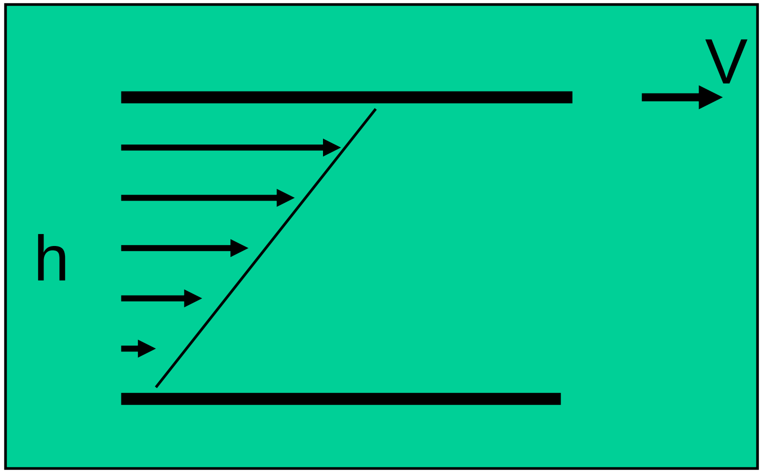
Chapter 6

Viscosity, etc.

Hooke's law: in **solid**, stress is proportional to strain

Fluid continuously deforms under an applied shear stress

Newtonian fluid: stress is linearly proportional to time rate of strain



$$\tau \propto \frac{V}{h}$$

$$\tau = \mu \frac{\partial u}{\partial y}$$

Water, air are Newtonian; blood is non-Newtonian

Chapter 6

Stokes Relations

Extended Newtonian idea to multi-dimensional flows

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} + \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)$$

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

$$\tau_{yy} = 2\mu \frac{\partial v}{\partial y} + \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)$$

$$\tau_{xz} = \tau_{zx} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)$$

$$\tau_{zz} = 2\mu \frac{\partial w}{\partial z} + \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)$$

$$\tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)$$

Chapter 6

Stokes Hypothesis

Choose λ so that normal stresses sum to zero

$$\tau_{xx} + \tau_{yy} + \tau_{zz} = 0$$

$$\lambda = -\frac{2}{3}\mu$$

Chapter 6

Conservation of momentum with viscosity

$$-\frac{dp}{dx}+\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}=\frac{\partial(\rho u^2)}{\partial x}+\frac{\partial(\rho vu)}{\partial y}+\frac{\partial(\rho wu)}{\partial z}+\frac{\partial(\rho u)}{\partial t}$$

$$-\frac{dp}{dy}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yz}}{\partial z}=\frac{\partial(\rho uv)}{\partial x}+\frac{\partial(\rho v^2)}{\partial y}+\frac{\partial(\rho wv)}{\partial z}+\frac{\partial(\rho v)}{\partial t}$$

$$-\frac{dp}{dz}+\frac{\partial\tau_{zz}}{\partial z}+\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}=\frac{\partial(\rho uw)}{\partial x}+\frac{\partial(\rho vw)}{\partial y}+\frac{\partial(\rho w^2)}{\partial z}+\frac{\partial(\rho w)}{\partial t}$$

Chapter 6

Incompressible, Steady 2-D flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial x} = \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial y} = \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$

Kinematic viscosity $\nu = \dfrac{\mu}{p}$

Chapter 6

2D Euler Equations – no viscosity

If incompressible

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

$$-\frac{dp}{dx} = \frac{d(\rho u^2)}{dx} + \frac{d(\rho uv)}{dy}$$

$$-\frac{dp}{dy} = \frac{d(\rho uv)}{dx} + \frac{d(\rho v^2)}{dy}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

$$-\frac{1}{\rho}\frac{dp}{dx} = u\frac{du}{dx} + v\frac{du}{dy}$$

$$-\frac{1}{\rho}\frac{dp}{dy} = u\frac{dv}{dx} + v\frac{dv}{dy}$$

Chapter 6

2D Equations review

$$\frac{\partial (\rho)}{\partial t} = -\left(\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} \right)$$

$$\frac{\partial (\rho u)}{dt} = -\left(\frac{\partial (\rho u^2)}{dx} + \frac{\partial (\rho uv)}{dy} + \frac{\partial (p)}{dx} \right)$$

$$\frac{\partial (\rho v)}{dt} = -\left(\frac{\partial (\rho uv)}{dx} + \frac{\partial (\rho v^2)}{dy} + \frac{\partial (p)}{dy} \right)$$