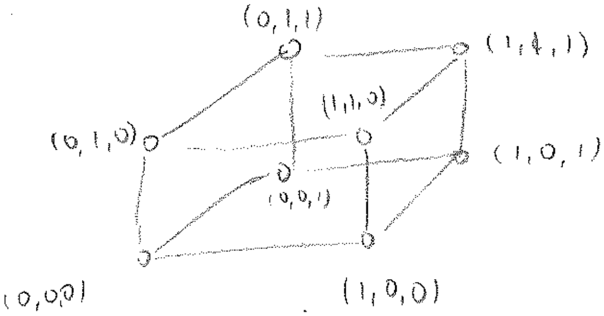
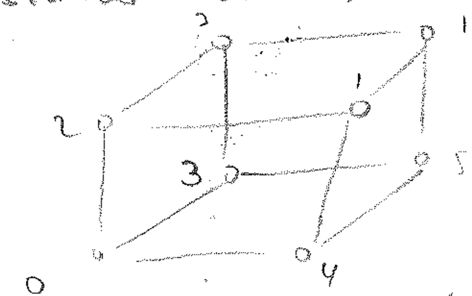
Interpolation

interpolate trilinearly at location $x = \begin{pmatrix} 0.3 \\ 0.2 \\ 0.4 \end{pmatrix}$ within the cell given as



the scalar values at the vertices are defined as follows:



Solution:

① $0.7 \cdot 3 + 0.3 \cdot 1 = 2.1 + 0.3 = 2.4$

$0.7 \cdot 2 + 0.3 \cdot 1 = 1.4 + 0.3 = 1.7$

$0.7 \cdot 3 + 0.3 \cdot 5 = \cancel{2.1} + 1.5 = \cancel{8}\ 3.6$

$0.7 \cdot 0 + 0.3 \cdot 4 = 0 + 1.2 = 1.2$

② $0.8 \cdot 1.7 + 0.4 \cdot 2.4 = \cancel{1.84}\ 1.98$

$0.8 \cdot 1.2 + 0.4 \cdot 3.6 = \cancel{1.56}\ \cancel{1.68}\ 2.16$

③ $\cancel{0.8 \cdot 1.68 + 0.2 \cdot 1.7} = \cancel{1.856}\ 1.688$

$0.8 \cdot 2.16 + 0.2 \cdot 1.98 = \cancel{2.664}\ 2.124$

$\Rightarrow$ the interpolated value of x is ~~1.856~~ 2.124

interpolate inside a triangle at location $x = \begin{pmatrix}1.2\\0.7\end{pmatrix}$. The triangle has the three vertices $t_1 = \begin{pmatrix}0\\1\end{pmatrix}$, $t_2 = \begin{pmatrix}1\\2\end{pmatrix}$, $t_3 = \begin{pmatrix}2\\0\end{pmatrix}$ and scalar values $v_1 = 1$, $v_2 = 3$, $v_3 = 2$ where v_i corresponds to t_i.

Solution:

Compute the barycentric coordinates:

$$\left.\begin{array}{l} b_1 + b_2 + b_3 = 1 \\ t_1 b_1 + t_2 b_2 + t_3 b_3 = x \end{array}\right\} \Rightarrow \begin{array}{rcll} b_1 + b_2 + b_3 &=& 1 & (1) \\ b_2 + 2b_3 &=& 1.2 & (2) \\ b_1 + 2b_2 &=& 0.7 & (3) \end{array}$$

$(1) - (2)$: $-b_2 + b_3 = 0.3$

$+(2)$: $3b_3 = 1.5 \Rightarrow b_3 = 0.5$

$(2) \Rightarrow b_2 + 1 = 1.2 \Rightarrow b_2 = 0.2$

$(3) \Rightarrow b_1 + 0.4 = 0.7 \Rightarrow b_1 = 0.3$

Alternative: use Cramer's rule:

$$Dx = d \Rightarrow D = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}, \quad d = \begin{pmatrix} 1 \\ 1.2 \\ 0.7 \end{pmatrix}$$

$$\det D = |D| = \det\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} + \det\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = -4 + 2 - 1 = -3$$

$$D_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1.2 & 1 & 2 \\ 0.7 & 2 & 0 \end{pmatrix}; \quad D_2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1.2 & 2 \\ 1 & 0.7 & 0 \end{pmatrix}; \quad D_3 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1.2 \\ 1 & 2 & 0.7 \end{pmatrix}$$

$$\det D_1 = \det\begin{pmatrix} 1.2 & 1 \\ 0.7 & 2 \end{pmatrix} - 2\det\begin{pmatrix} 1 & 1 \\ 0.7 & 2 \end{pmatrix}$$

$$= 2.4 - 0.7 - 4 + 1.4 = -0.9$$

$$\det D_2 = \det\begin{pmatrix} 1.2 & 2 \\ 0.7 & 0 \end{pmatrix} + \det\begin{pmatrix} 1 & 1 \\ 1.2 & 2 \end{pmatrix} = -1.4 + 2 - 1.2 = -0.6$$

$$\det D_3 = \det\begin{pmatrix} 1 & 1.2 \\ 2 & 0.7 \end{pmatrix} + \det\begin{pmatrix} 1 & 1 \\ 1 & 1.2 \end{pmatrix} = 0.7 - 2.4 + 1.2 - 1 = -1.5$$

$$\Rightarrow b_1 = \frac{-0.9}{-3} = 0.3; \quad b_2 = \frac{-0.6}{-3} = 0.2; \quad b_3 = \frac{-1.5}{-3} = 0.5$$

Hence, the interpolated value can be computed as:

$$0.3 \cdot 1 + 0.2 \cdot 3 + 0.5 \cdot 2 = 0.3 + 0.6 + 1 = \underline{1.9}$$

4.

What is the difference between a path line and a stream-line? Do they describe a similar curve? Why or why not?

Solution: The particle traced for generating a path line (usually dye) can have a mass. Due to inertia the curves may be different.

Tensors:

Let $t = \begin{pmatrix} 0 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix}$ be a given tensor. Compute the eigenvalues and eigenvectors of this tensor.

Solution:

$$\chi(\lambda) = |\lambda I - t| = \begin{vmatrix} \lambda & -\sqrt{3} \\ -\sqrt{3} & \lambda - 2 \end{vmatrix} = \lambda(\lambda - 2) - 3 = \lambda^2 - 2\lambda - 3$$

$$= (\lambda + 1)(\lambda - 3)$$

$$\Rightarrow \lambda_1 = 1, \quad \lambda_2 = 3$$

We are looking for vectors e_1, e_2 such that

$(\lambda_i I - A)\, e_i = 0$

$$i=1: \begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = s \cdot \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \Rightarrow e_i = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$$

$x - \sqrt{3}y = 0 \Leftrightarrow x = \sqrt{3}y \Leftrightarrow y = \sqrt{3}x$

$$i=2: \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = s \cdot \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix} \Rightarrow e_2 = \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}$$

$-\sqrt{3}x + y = 0 \Leftrightarrow x = -\sqrt{3}y$

$$A = \begin{pmatrix} 3 & 1 & -4 \\ 1 & 3 & -4 \\ -4 & -4 & 8 \end{pmatrix}$$

$$\chi(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda-3 & -1 & 4 \\ -1 & \lambda-3 & 4 \\ 4 & 4 & \lambda-8 \end{vmatrix} = (\lambda-3)\begin{vmatrix} \lambda-3 & 4 \\ 4 & \lambda-8 \end{vmatrix} + \begin{vmatrix} -1 & 4 \\ 4 & \lambda-8 \end{vmatrix} + 4\begin{vmatrix} -1 & \lambda-3 \\ 4 & 4 \end{vmatrix}$$

$$= (\lambda-3)(\lambda-3)(\lambda-8) - (\lambda-3)16 - (\lambda-8) - 16 - 16 - 16(\lambda-3)$$
$$= (\lambda^2 - 6\lambda + 9)(\lambda-8) - 16\lambda + 48 - \lambda + 8 - 32 - 16\lambda + 48$$
$$= \lambda^3 - 6\lambda^2 + 9\lambda - 8\lambda^2 + 48\lambda - 72 - 23\lambda + 72$$
$$= \lambda^3 - 14\lambda^2 + 24\lambda$$
$$= \lambda(\lambda^2 - 14\lambda + 24)$$
$$= \lambda(\lambda-2)(\lambda-12)$$

$$\Rightarrow \lambda_1 = 0, \ \lambda_2 = 2, \ \lambda_3 = 12$$

$$A \cdot e_i = \lambda_i e_i$$

$i=0$: $$\begin{pmatrix} 3 & 1 & -4 \\ 1 & 3 & -4 \\ -4 & -4 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Leftrightarrow \begin{array}{l} 3x + y - 4z = 0 \\ 4x + 12y - 16z = 0 \\ -4x - 4y + 8z = 0 \end{array}$$

$$\Leftrightarrow \begin{array}{l} 3x + y - 4z = 0 \\ 4x + 12y - 16z = 0 \\ 8y - 8z = 0 \end{array} \Leftrightarrow \begin{array}{l} 3x + y - 4z = 0 \\ 3x + 9y - 12z = 0 \\ y - z = 0 \end{array} \Leftrightarrow \begin{array}{l} 3x + y - 4z = 0 \\ 8y - 16z = 0 \\ y - z = 0 \end{array}$$

$$\Leftrightarrow \begin{array}{l} 3x + y - 4z = 0 \\ y - 2z = 0 \\ y - z = 0 \end{array} \Leftrightarrow \begin{array}{l} 3x + y - 4z = 0 \\ y - 2z = 0 \\ z = 0 \end{array} \Leftrightarrow \begin{array}{l} 3x - 2z = 0 \\ y - 2z = 0 \\ z = 0 \end{array}$$

$$\Leftrightarrow \begin{array}{l} x - \frac{2}{3}z = 0 \\ y - 2z = 0 \\ z = 0 \end{array} \Rightarrow x = y = z = 0$$

$i=1$: $$\begin{pmatrix} 1 & 1 & -4 \\ 1 & 1 & -4 \\ -4 & -4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Leftrightarrow \begin{array}{l} x + y - 4z = 0 \\ x + y - 4z = 0 \\ -4x - 4y + 6z = 0 \end{array} \Leftrightarrow \begin{array}{l} 0 = 0 \\ x + y - 4z = 0 \\ 10z = 0 \end{array}$$

$$\begin{aligned} 3x + y - 4z &= 0 \\ 4x + 12y - 16z &= 0 \\ -4x - 4y + 8z &= 0 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} x + \tfrac{1}{3}y - 4z &= 0 \\ x + 3y - 4z &= 0 \\ x + y - 2z &= 0 \end{aligned} \quad \Leftrightarrow$$

Why are tricycle ships better?

What is the underlying principle of LIC?

When is mesh decimation beneficial?