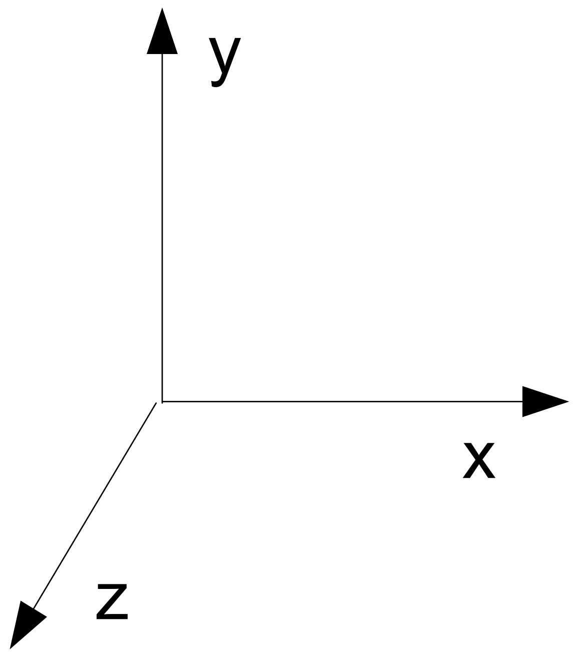
Addendum

Vector Review

Coordinate Systems

Right handed coordinate system



Vector Arithmetic

$$\mathbf{a} = \begin{bmatrix} a_x & a_y & a_z \end{bmatrix}$$
$$\mathbf{b} = \begin{bmatrix} b_x & b_y & b_z \end{bmatrix}$$
$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} a_x + b_x & a_y + b_y & a_z + b_z \end{bmatrix}$$
$$\mathbf{a} - \mathbf{b} = \begin{bmatrix} a_x - b_x & a_y - b_y & a_z - b_z \end{bmatrix}$$
$$-\mathbf{a} = \begin{bmatrix} -a_x & -a_y & -a_z \end{bmatrix}$$
$$s\mathbf{a} = \begin{bmatrix} sa_x & sa_y & sa_z \end{bmatrix}$$

Vector Magnitude

The magnitude (length) of a vector is:

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

A vector with length=1.0 is called a *unit vector*

We can also *normalize* a vector to make it a unit vector:

$$\frac{\mathbf{v}}{|\mathbf{v}|}$$

Dot Product

$$\mathbf{a} \cdot \mathbf{b} = \sum a_i b_i$$

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z$$

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

Dot Product

$$\mathbf{a}\cdot\mathbf{b}=\sum a_i b_i$$

$$\mathbf{a}\cdot\mathbf{b}=a_x b_x + a_y b_y + a_z b_z$$

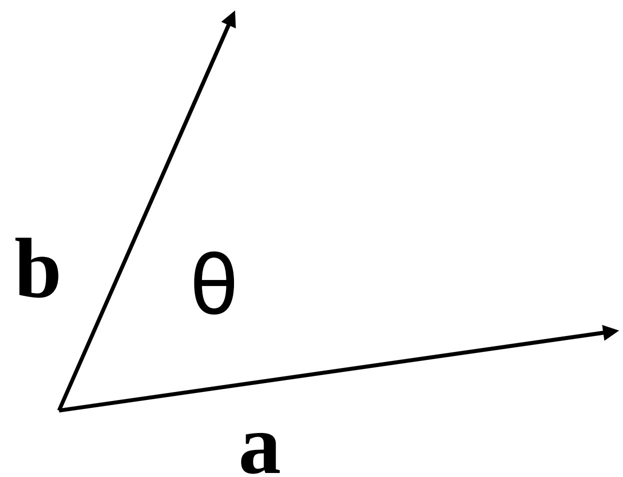
$$\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta$$

$$\mathbf{a}\cdot\mathbf{b}=\mathbf{a}^T\mathbf{b}$$

$$\mathbf{a}\cdot\mathbf{b}=\begin{bmatrix}a_x & a_y & a_z\end{bmatrix}\begin{bmatrix}b_x\\ b_y\\ b_z\end{bmatrix}$$

Example: Angle Between Vectors

How do you find the angle θ between vectors **a** and **b**?

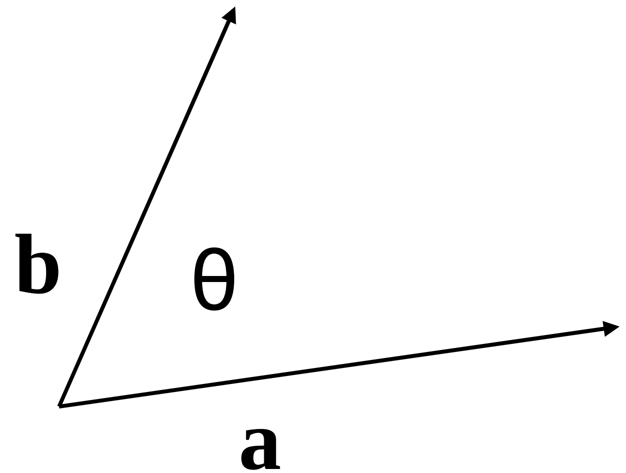


Example: Angle Between Vectors

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

$$\cos\theta = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right)$$

$$\theta = \cos^{-1}\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right)$$



Dot Products with General Vectors

The dot product is a scalar value that tells us something about the relationship between two vectors

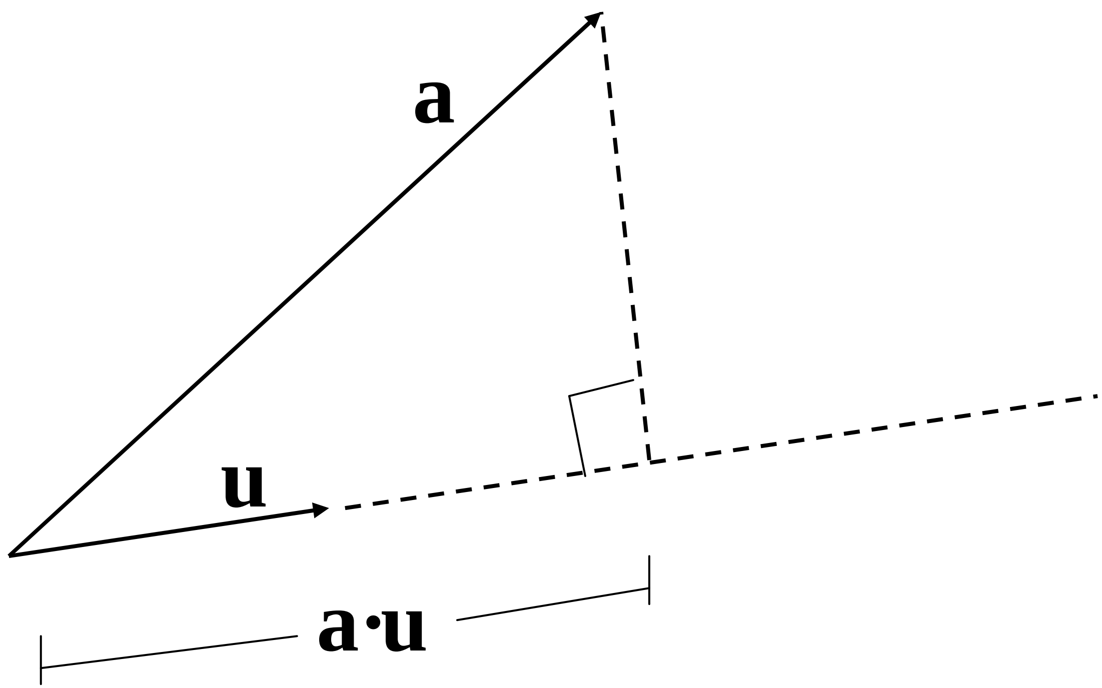
If $\mathbf{a} \cdot \mathbf{b} > 0$ then $\theta < 90^\circ$

If $\mathbf{a} \cdot \mathbf{b} < 0$ then $\theta > 90^\circ$

If $\mathbf{a} \cdot \mathbf{b} = 0$ then $\theta = 90^\circ$ (or one or more of the vectors is degenerate (0,0,0))

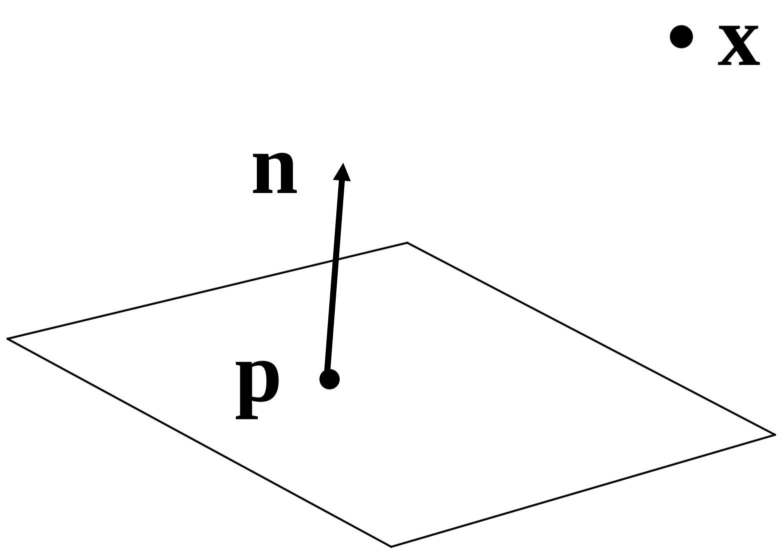
Dot Products with One Unit Vector

If |**u**|=1.0 then **a**·**u** is the length of the *projection* of **a** onto **u**



Example: Distance to Plane

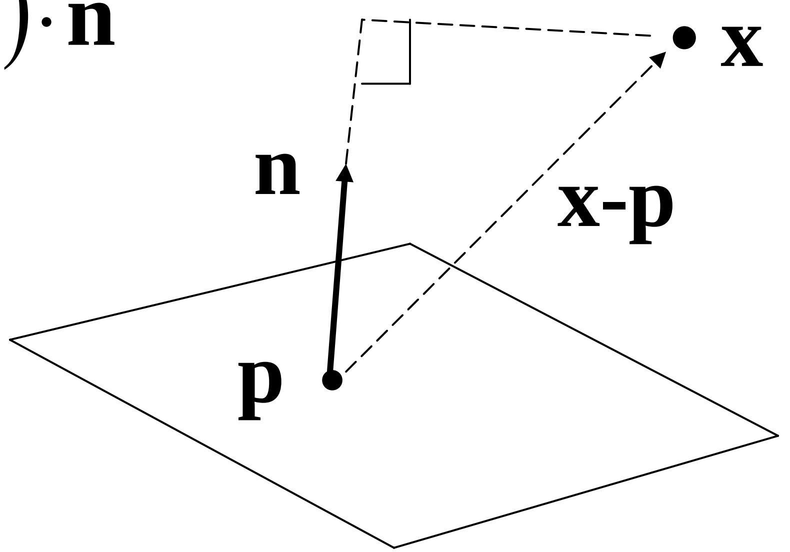
A plane is described by a point **p** on the plane and a unit normal **n**. Find the distance from point **x** to the plane



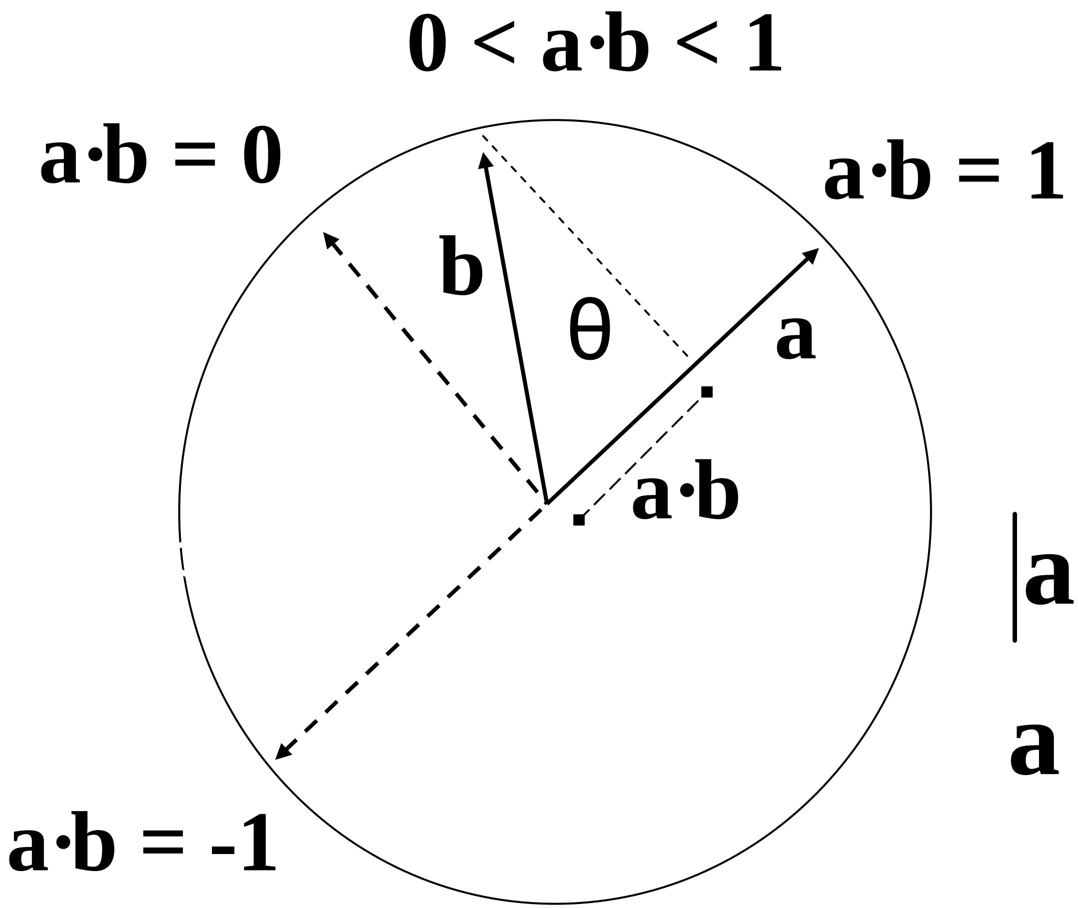
Example: Distance to Plane

The distance is the length of the projection of **x-p** onto **n**:

$$dist = (\mathbf{x} - \mathbf{p}) \cdot \mathbf{n}$$



Dot Products with Unit Vectors



$$|\mathbf{a}| = |\mathbf{b}| = 1.0$$

$$\mathbf{a} \cdot \mathbf{b} = \cos(\theta)$$

Cross Product

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_y b_z - a_z b_y & a_z b_x - a_x b_z & a_x b_y - a_y b_x \end{bmatrix}$$

Properties of the Cross Product

$\mathbf{a} \times \mathbf{b}$ is a *vector* perpendicular to both **a** and **b**, in the direction defined by the right hand rule

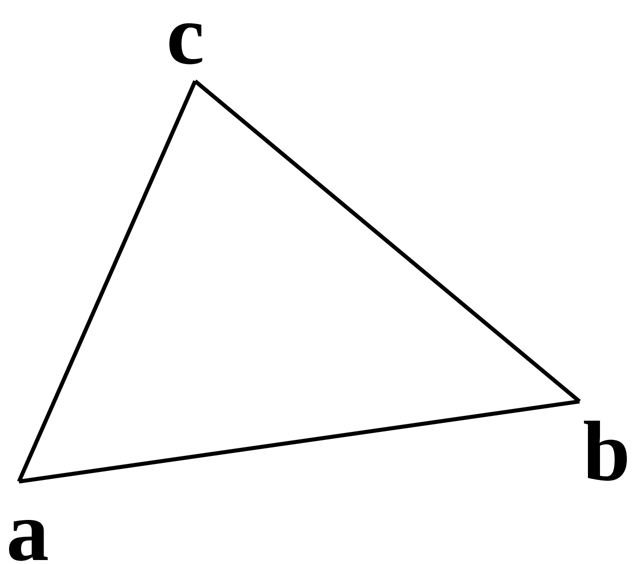
$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$$

$|\mathbf{a} \times \mathbf{b}| =$ area of parallelogram **ab**

$|\mathbf{a} \times \mathbf{b}| = 0$ if **a** and **b** are parallel

Example: Normal of a Triangle

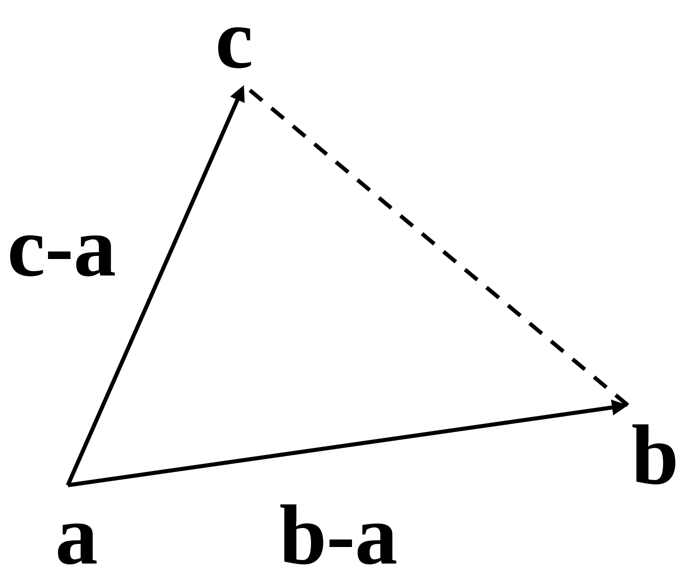
Find the unit length normal of the triangle defined by 3D points **a**, **b**, and **c**



Example: Normal of a Triangle

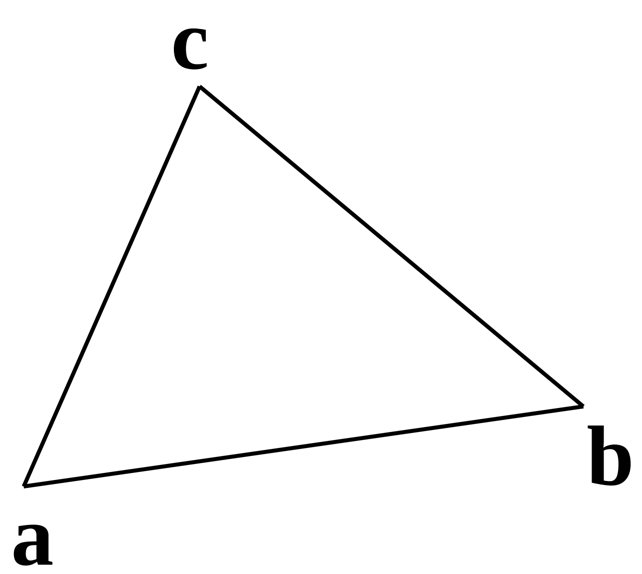
$$\mathbf{n}^* = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$$

$$\mathbf{n} = \frac{\mathbf{n}^*}{\left|\mathbf{n}^*\right|}$$



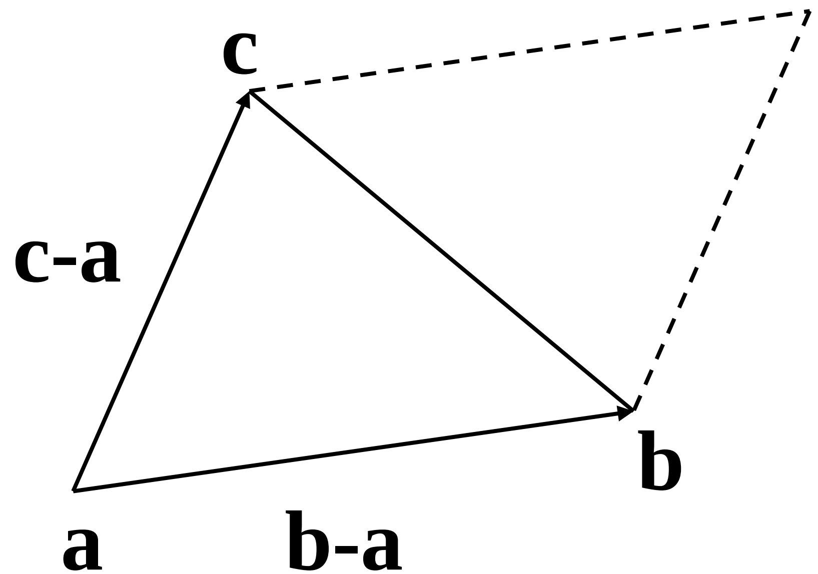
Example: Area of a Triangle

Find the area of the triangle defined by 3D points **a**, **b**, and **c**



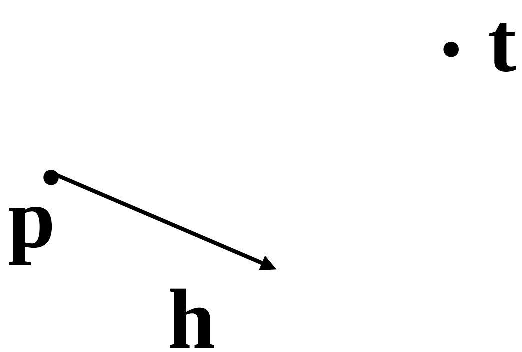
Example: Area of a Triangle

$$area = \frac{1}{2}\left|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})\right|$$



Example: Alignment to Target

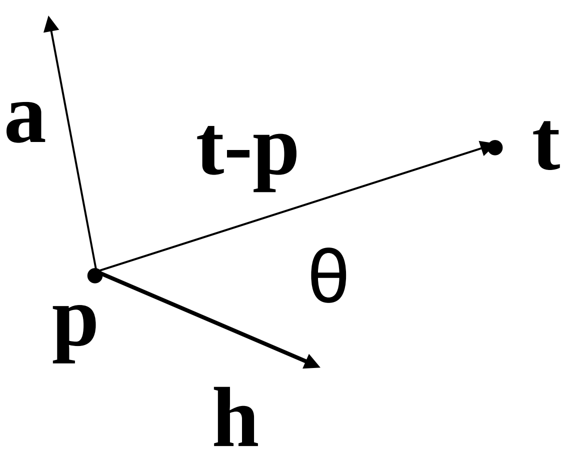
An object is at position **p** with a unit length heading of **h**. We want to rotate it so that the heading is facing some target **t**. Find a unit axis **a** and an angle θ to rotate around.



Example: Alignment to Target

$$\mathbf{a} = \frac{\mathbf{h} \times (\mathbf{t} - \mathbf{p})}{\left|\mathbf{h} \times (\mathbf{t} - \mathbf{p})\right|}$$

$$\theta = \cos^{-1}\left(\frac{\mathbf{h} \cdot (\mathbf{t} - \mathbf{p})}{\left|(\mathbf{t} - \mathbf{p})\right|}\right)$$



Addendum

Cramer’s Rule

This is a great method.

Cramer’s Rule is a neat way to evaluate systems and if you put the work in now you’ll do fine. It can be used for any size (2 by 2, 3 by 3 or even larger) system.

It is easy to memorize and fast.

I’m going to show you where Cramer’s Rule comes from; but first, some definitions

Definitions

Determinant – a square array

2^{nd} Order Determinant – a 2 by 2 array

3^{rd} Order Determinant – a 3 by 3 array

Elements – The things in the array

What does a determinant look like?

A 2nd order determinant looks like this

$$\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

And the value of the determinant =

Diagonal down right – diagonal down left $\quad ae - bd$

Examples

Evaluate

1. $$\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}$$

2. $$\begin{vmatrix} 2 & -2 \\ 6 & 1 \end{vmatrix}$$

Why is this useful for systems?

Lets work through an elimination example using all variables; then we can see how the determinant will be useful in solving.

$$ax + by = c$$
$$dx + ey = f$$

$\xrightarrow[\text{eliminate } y]{\text{Lets}}$

$$aex + bey = ce$$
$$bdx + bey = bf$$

$$aex - bdx = ce - bf$$
$$x(ae - bd) = ce - bf$$

$$x = \frac{ce - bf}{ae - bd} \longrightarrow x = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}$$

Look familiar?

If you apply the same process but eliminate x

$$y = \frac{af - cd}{ae - bd} \longrightarrow y = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}$$

So, what does Cramer's Rule say?

Cramer's Rule

Given a system

$$ax + by = c$$
$$dx + ey = f$$

Replace solutions in x column to solve for x

Replace solutions in y column to solve for y

$$x = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}$$

Denominators are coefficient determinants

What do you think is the trick?

Examples

Solve using Cramer's Rule

1.

$$6x + 7y = -9$$

$$x - y = 5$$

2.

$$5x + 4b = -1$$

$$2x - b = 10$$

Cramer's Rule in General

Cramer's Rule: For the system of equations **Ax** = **y,** where A is an *nxn* nonsingular matrix, the solution for the *i*th endogenous variable, x_i, is

$$\mathbf{x_i = |A_i|/|A|}$$

where the matrix $\mathbf{A_i}$ represents a matrix that is identical to the matrix **A** but for the replacement of the *i*th column with the *nx1* vector **y**.

Linear Systems in Matrix Form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad (1)$$

Solution of Linear Systems

Each side of the equation

$$A \cdot \vec{x} = \vec{b} \quad (2)$$

Can be multiplied by A^{-1} :

$$A^{-1} A \cdot \vec{x} = A^{-1} \vec{b}$$

Due to the definition of A^{-1}: $A^{-1} A \cdot \vec{x} = I \cdot \vec{x} = \vec{x}$

Therefore the solution of (2) is:

$$\vec{x} = A^{-1} \vec{b}$$

Consistency (Solvability)

- A^{-1} does not exist for every A
- The linear system of equations **A·x=b** has a solution, or said to be consistent IFF

$$\mathrm{Rank}\{A\}=\mathrm{Rank}\{A|b\}$$

- A system is inconsistent when

$$\mathrm{Rank}\{A\}<\mathrm{Rank}\{A|b\}$$

Rank{A} is the maximum number of linearly independent columns or rows of A. Rank can be found by using ERO (Elementary Row Oparations) or ECO (Elementary column operations).

Elementary row and column operations

The following operations applied to the augmented matrix [A|b], yield an equivalent linear system

- Interchanges: The order of two rows/columns can be changed
- Scaling: Multiplying a row/column by a nonzero constant
- Sum: The row can be replaced by the sum of that row and a nonzero multiple of any other row.

One can use ERO and ECO to find the Rank as follows:

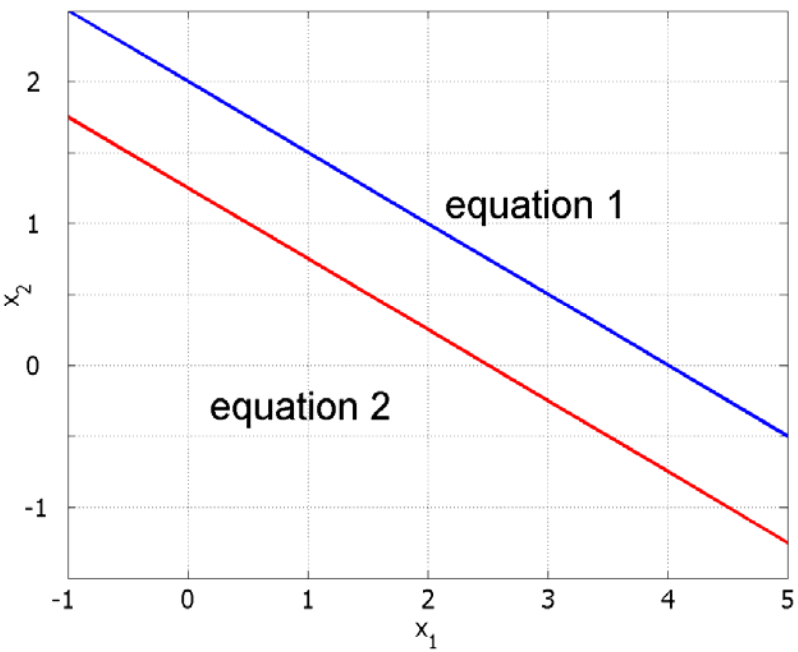
ERO$\Rightarrow$minimum # of rows with at least one nonzero entry

or

ECO$\Rightarrow$minimum # of columns with at least one nonzero entry

An inconsistent example: Geometric interpretation

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & -3 \end{bmatrix}$$

Uniqueness of solutions

- The system has a unique solution IFF

  Rank{A}=Rank{A|b}=n

  n is the order of the system

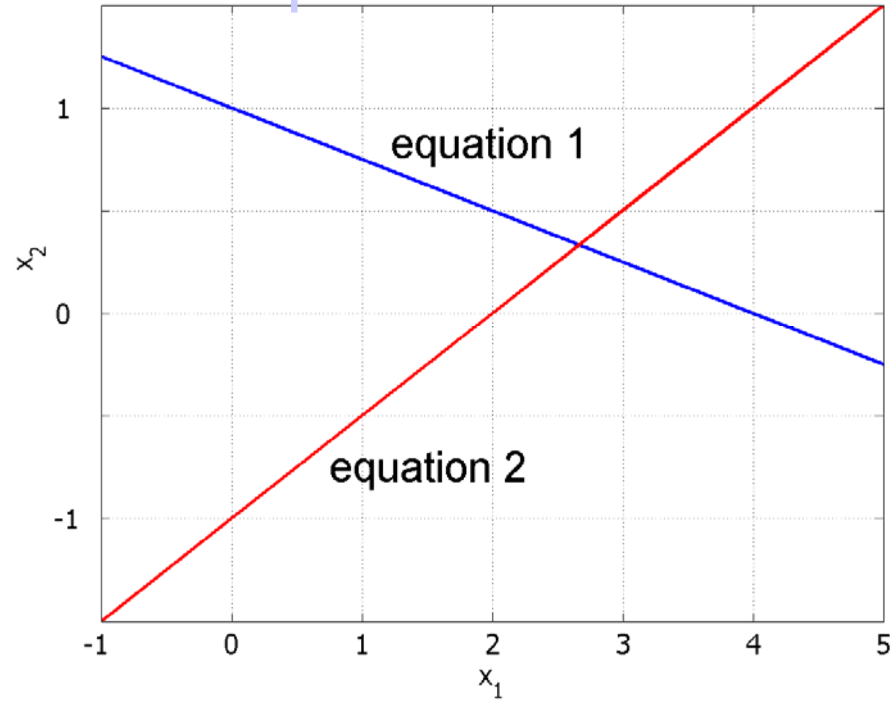
- Such systems are called full-rank systems

Full-rank systems

- If Rank{A}=n

  Det{A} $\neq$ 0 $\Rightarrow$ A^{-1} exists $\Rightarrow$ Unique solution

$$\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$



Rank deficient matrices

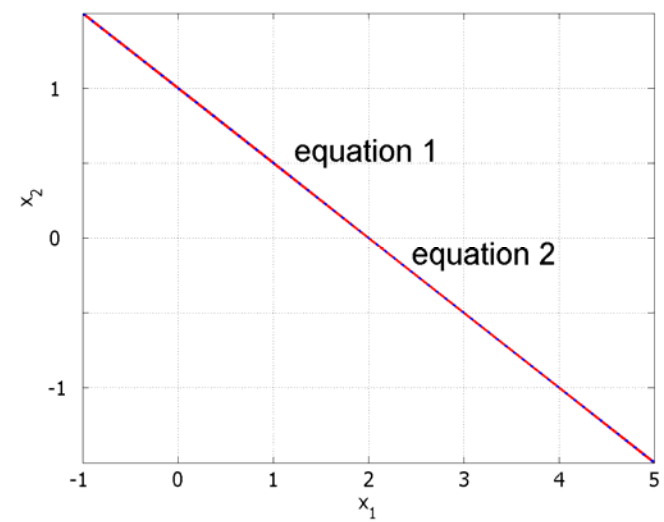
- If Rank{A}=m<n

  Det{A} = 0 $\Rightarrow$ A is singular so not invertible

  infinite number of solutions (n-m free variables)

  under-determined system

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$



Ill-conditioned system of equations

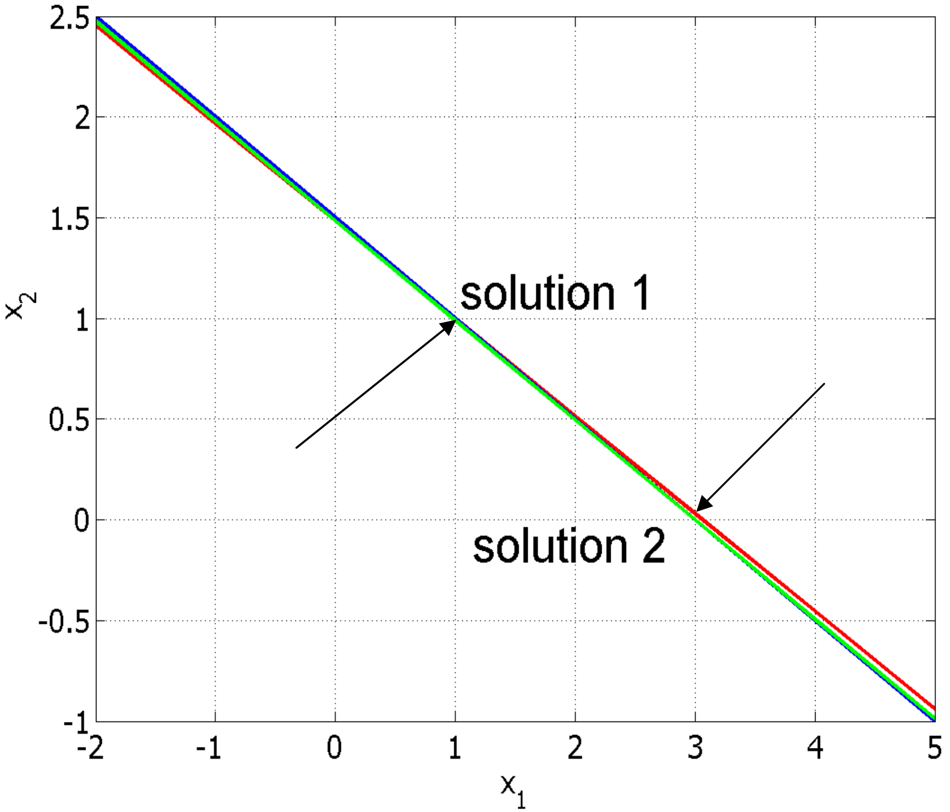
- A small deviation in the entries of A matrix, causes a large deviation in the solution.

$$\begin{bmatrix} 1 & 2 \\ 0.48 & 0.99 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1.47 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0.49 & 0.99 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1.47 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Ill-conditioned continued.....

- A linear system of equations is said to be “ill-conditioned” if the coefficient matrix tends to be singular



Gaussian Elimination

- By using ERO, matrix A is transformed into an upper triangular matrix (all elements below diagonal 0)
- Back substitution is used to solve the upper-triangular system

$$
\begin{bmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ii} & \cdots & a_{in} \\ \vdots & & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix} \underset{\text{ERO}}{\Rightarrow} \begin{bmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & \tilde{a}_{ii} & \cdots & \tilde{a}_{in} \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \tilde{a}_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ \tilde{b}_i \\ \vdots \\ \tilde{b}_n \end{bmatrix} \Big\uparrow
$$

Back substitution

Pivotal Element

Pivotal element

$$\begin{bmatrix} \boxed{a_{11}^{(1)}} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2n}^{(1)} \\ a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} & \cdots & a_{3n}^{(1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1}^{(1)} & a_{n2}^{(1)} & a_{n3}^{(1)} & \cdots & a_{nn}^{(1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(1)} \\ b_3^{(1)} \\ \vdots \\ b_n^{(1)} \end{bmatrix}$$

The first coefficient of the first row (pivot) is used to zero out first coefficients of other rows. In terms of numerical stability it is usually best to use the largest element in the column, i.e. $\max_{k \le j \le n} \left| a_{jk}^{(k)} \right|$

First step of elimination

$$
\begin{aligned}
m_{2,1} &= a_{21}^{(1)} / a_{11}^{(1)} \\
m_{3,1} &= a_{31}^{(1)} / a_{11}^{(1)} \\
&\vdots \\
m_{n,1} &= a_{n1}^{(1)} / a_{11}^{(1)}
\end{aligned}
\qquad
\begin{bmatrix}
a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} \\
0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\
0 & a_{32}^{(2)} & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & a_{n2}^{(2)} & a_{n3}^{(2)} & \cdots & a_{nn}^{(2)}
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n
\end{bmatrix}
=
\begin{bmatrix}
b_1^{(1)} \\ b_2^{(2)} \\ b_3^{(2)} \\ \vdots \\ b_n^{(2)}
\end{bmatrix}
$$

First row, multiplied by appropriate factor is subtracted from other rows.

Second step of elimination

Pivotal element

$$\begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & \boxed{a_{22}^{(2)}} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}^{(2)} & a_{n3}^{(2)} & \cdots & a_{nn}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(2)} \\ b_3^{(2)} \\ \vdots \\ b_n^{(2)} \end{bmatrix}$$

Second step of elimination

$$
\begin{aligned}
m_{3,2} &= a_{32}^{(2)} / a_{22}^{(2)} \\
&\vdots \\
m_{n,2} &= a_{n2}^{(2)} / a_{22}^{(2)}
\end{aligned}
\qquad
\begin{bmatrix}
a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} \\
0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\
0 & 0 & a_{33}^{(3)} & \cdots & a_{3n}^{(3)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & a_{n3}^{(3)} & \cdots & a_{nn}^{(3)}
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n
\end{bmatrix}
=
\begin{bmatrix}
b_1^{(1)} \\ b_2^{(2)} \\ b_3^{(3)} \\ \vdots \\ b_n^{(3)}
\end{bmatrix}
$$

Gaussion elimination algorithm

Define number of steps as p (pivotal row)

For p=1,n-1

For r=p+1 to n

$$m_{r,p} = a_{rp}^{(p)} / a_{pp}^{(p)}$$

$$a_{rp}^{(p)} = 0$$

$$b_r^{(p+1)} = b_r^{(p)} - m_{r,p} \times b_p^{(p)}$$

$$a_{rc}^{(p+1)} = a_{rc}^{(p)} - m_{r,p} \times a_{pc}^{(p)}$$

Back substitution algorithm

$$\begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & 0 & a_{33}^{(3)} & \cdots & a_{3n}^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a_{n-1\,n-1}^{(n)} & a_{n-1\,n}^{(n)} \\ 0 & 0 & 0 & 0 & a_{nn}^{(n)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(2)} \\ b_3^{(3)} \\ \vdots \\ b_{n-1}^{(n-1)} \\ b_n^{(n)} \end{bmatrix}$$

Back substitution algorithm

The answer is obtained as following:

$$x_n = \frac{b_n^{(n)}}{a_{nn}^{(n)}}$$

$$x_{n-1} = \frac{1}{a_{n-1n-1}^{(n-1)}}\left[b_{n-1}^{(n-1)} - a_{n-1n}^{n-1} x_n\right]$$

$$x_i = \frac{1}{a_{ii}^{(i)}}\left[b_i^{(i)} - \sum_{k=i+1}^{n} a_{ik}^{(i)} x_k\right] \quad i = n-1, n-2, \ldots, 1$$

LU decomposition

If we now define a matrix R

$$R = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & 0 & a_{33}^{(3)} & \cdots & a_{3n}^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a_{n-1\,n-1}^{(n)} & a_{n-1\,n}^{(n)} \\ 0 & 0 & 0 & 0 & a_{nn}^{(n)} \end{bmatrix}$$

and another matrix $L = (l_{ij})$ with

$$l_{ik} = \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}}$$

then we get: $A = L \cdot R$

LU decomposition

Note that the matrices R and L are upper and lower triangular matrices. Hence, we can solve the linear equation system in two steps:

$$L\,c = b$$

$$R\,x = c$$

Solving these two systems can be achieved similar to the using a similar algorithm we used for back substitution.

Also note that solving the system of linear equations for different solutions b does not require a repetition of the Gaussian elimination algorithm.

Cholesky decomposition

If the matrix A is symmetric, a solution for the LU decomposition is even easier since there exist the following decomposition for those matrices:

$$A = C^T C$$

where

$$C = \begin{pmatrix} c_{11} & \cdots & \cdots & c_{1n} \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & c_{nn} \end{pmatrix}$$

Cholesky decomposition

How can we compute C? Let us look at the equation:

$$\begin{pmatrix} c_{11} & 0 & \cdots & 0 \\ c_{12} & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{pmatrix} \begin{pmatrix} c_{11} & \cdots & \cdots & c_{1n} \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & c_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix}$$

Hence:

$$a_{kk} = \sum_{j=1}^{k} c_{jk} c_{jk} \Rightarrow c_{kk} = \sqrt{a_{kk} - \sum_{j=1}^{k-1} c_{jk}^2}$$

$$a_{kl} = \sum_{j=1}^{k} c_{jk} c_{jl} \Rightarrow c_{kl} = \frac{a_{kl} - \sum_{j=1}^{k-1} c_{jk} c_{jl}}{c_{kk}} \quad \text{for} \quad k < l \le n$$