Find the cubic polynomial that interpolates the following points

$$p_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad p_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad p_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \qquad p_3 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

- by solving the linear equation system:

cubic polynomial: $p(x) = ax^3 + bx^2 + cx + d$

linear equation system:

$$\begin{aligned} p(0) = 1 &\Rightarrow d = 1 \\ p(1) = 2 &\Rightarrow a + b + c + d = 2 \\ p(3) = 5 &\Rightarrow 27a + 9b + 3c + d = 5 \\ p(4) = 3 &\Rightarrow 64a + 16b + 4c + d = 3 \end{aligned}$$

plugging in $d = 1$ gives us:

$$\begin{aligned} a + b + c &= 1 \\ 27a + 9b + 3c &= 4 \\ 64a + 16b + 4c &= 2 \end{aligned} \iff \begin{aligned} a + b + c &= 1 \\ 24a + 6b &= 1 \\ 60a + 12b &= -2 \end{aligned} \iff \begin{aligned} a + b + c &= 1 \\ 24a + 6b &= 1 \\ 12a &= -4 \end{aligned}$$

$$\iff \begin{aligned} a &= -1/3 \\ 6b &= 9 \\ b + c &= 1\tfrac{1}{3} \end{aligned} \iff \begin{aligned} a &= -1/3 \\ b &= \tfrac{3}{2} \\ c &= 1\tfrac{1}{3} - b = \tfrac{4}{3} - \tfrac{3}{2} = -\tfrac{1}{6} \end{aligned} \Rightarrow a = -1/3,\ b = 3/2,\ c = -1/6,\ d = 1$$

$$\Rightarrow p(x) = -\tfrac{1}{3}x^3 + 3/2\, x^2 - 1/6\, x + 1$$

- by using Lagrange polynomials:

$$L_i(t) = \prod_{k=0;\,k\neq i}^{n} \frac{t - t_k}{t_i - t_k}$$

First, compute the basis polynomials:

$t_0 = 0,\ t_1 = 1,\ t_2 = 3,\ t_3 = 4$

$$L_0(t) = \frac{t - t_1}{t_0 - t_1} \cdot \frac{t - t_2}{t_0 - t_2} \cdot \frac{t - t_3}{t_0 - t_3} = \frac{(t-1)\cdot(t-3)(t-4)}{(0-1)(0-3)(0-4)} = \frac{(t^2 - 4t + 3)(t-4)}{-12}$$

$$= -\frac{1}{12}(t^3 - 8t^2 + 19t - 12)$$

$$L_1(t) = \frac{t - t_0}{t_1 - t_0} \cdot \frac{t - t_2}{t_1 - t_2} \cdot \frac{t - t_3}{t_1 - t_3} = \frac{t\cdot(t-3)(t-4)}{(1-0)(1-3)(1-4)} = \frac{t\,(t^2 - 7t + 12)}{6}$$

$$= \frac{1}{6}(t^3 - 7t^2 + 12t)$$

$$L_2(t) = \frac{t - t_0}{t_2 - t_0} \cdot \frac{t - t_1}{t_2 - t_1} \cdot \frac{t - t_3}{t_2 - t_3} = \frac{t\cdot(t-1)(t-4)}{(3-0)(3-1)(3-4)} = \frac{t\cdot(t^2 - 5t + 4)}{-6}$$

$$= -\frac{1}{6}(t^3 - 5t^2 + 4t)$$

$$l_3(t) = \frac{t-t_0}{t_3-t_0} \cdot \frac{t-t_1}{t_3-t_1} \cdot \frac{t-t_2}{t_3-t_2} = \frac{t(t-1)(t-3)}{(4-0)(4-1)(4-3)} = \frac{t\cdot(t^2-4t+3)}{12}$$

$$= \frac{1}{12}(t^3 - 4t^2 + 3t)$$

Now we can plug in the given points:

$$p(x) = l_0(x) + 2\,l_1(x) + 5\,l_2(x) + 3\,l_3(x)$$

$$= -\frac{1}{12}t^3 + \frac{8}{12}t^2 - \frac{19}{12}t + 1 \qquad + \frac{1}{3}t^3 - \frac{7}{3}t^2 + 4t$$

$$- \frac{5}{6}t^3 + \frac{25}{6}t^2 - \frac{20}{6}t \qquad + \frac{1}{4}t^3 - t^2 + \frac{3}{4}t$$

$$= -\frac{4}{12}t^3 + \frac{18}{12}t^2 - \frac{2}{12}t + 1$$

$$= -2/3\,t^3 + 3/2\,t^2 - 1/6\,t + 1$$

Determine the Bézier control points such that the resulting Bézier polynomial interpolates the given points:

$t_0 = 0,\ t_1 = 1,\ t_2 = 3,\ t_3 = 4$

Bernstein basis:

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

$$p(t) = b_0 B_0^3(t) + b_1 B_1^3(t) + b_2 B_2^3(\ ) + b_3 B_3^3(t)$$

$$B_0^3(t) = (1-t)^3 = (1-t)(1-2t+t^2) = 1 - 3t - t^2 - t^3$$

$$B_1^3(t) = 3t(1-t)^2 = 3t(1-2t+t^2) = 3t - 6t^2 + 3t^3$$

$$B_2^3(t) = 3t^2(1-t) = 3t^2 - 3t^3$$

$$B_3^3(t) = t^3$$

$$\Rightarrow p(t) = b_0(1 - 3t^2 - t^3) + b_1(3t - 6t^2 + 3t^3) + b_2(3t^2 - 3t^3) + b_3 t^3$$

$p(0) = 1 \Rightarrow b_0 = 1$

$p(1) = 2 \Rightarrow -3b_0 + b_3 = 2$

$p(3) = 5 \Rightarrow -35b_0 + 36b_1 - 54b_2 + 27b_3 = 5$

$p(4) = 3 \Rightarrow -111b_0 + 96b_1 - 154b_2 + 64b_3 = 3$

$b_0 = 1$

$\Rightarrow b_3 = 2 + 3b_0 = 5$

$$\Rightarrow \begin{matrix} -35 + 36b_1 - 54b_2 + 135 = 5 \\ -111 + 96b_1 - 154b_2 + 320 = 3 \end{matrix} \Rightarrow \begin{matrix} 36b_1 - 54b_2 = -95 \\ 96b_1 - 154b_2 = -206 \end{matrix}$$

$36b_1 - 54b_2 = -95$

$$\Rightarrow \left(96 - \frac{36\cdot 154}{54}\right)b_1 = -206 + \frac{95\cdot 154}{54} \Rightarrow b_1 = \frac{-206 + \frac{95\cdot 154}{54}}{96 - \frac{36\cdot 154}{54}}$$

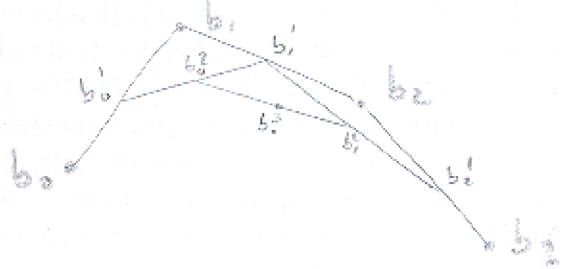
$$b_2 = \frac{95 + 36\cdot\frac{-206 + \frac{95\cdot 154}{54}}{96 - \frac{36\cdot 154}{54}}}{54}$$

Let $b(t)$ be a given Bézier segment defined by the control points $b_0 = \begin{pmatrix}1\\0\end{pmatrix}$, $b_1 = \begin{pmatrix}2\\2\end{pmatrix}$, $b_2 = \begin{pmatrix}3\\1\end{pmatrix}$, and $b_3 = \begin{pmatrix}4\\-1\end{pmatrix}$.

a) Assuming this Bézier segment is well-defined, i.e. there is a unique solution, of what degree n is the resulting polynomial?

The resulting polynomial is cubic, so $n = 3$.

b) Evaluate the Bézier segment at $t = 0.6$ (assume that the Bézier segment is defined for parameters $0 \le t \le 1$) using the de Casteljau algorithm.



$$b_0^1 = 0.4\,b_0 + 0.6\,b_1 = 0.4\begin{pmatrix}1\\0\end{pmatrix} + 0.6\begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}0.4 + 1.2\\0 + 1.2\end{pmatrix} = \begin{pmatrix}1.6\\1.2\end{pmatrix}$$

$$b_1^1 = 0.4\,b_1 + 0.6\,b_2 = 0.4\begin{pmatrix}2\\2\end{pmatrix} + 0.6\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}0.8 + 1.8\\0.8 + 0.6\end{pmatrix} = \begin{pmatrix}2.6\\1.4\end{pmatrix}$$

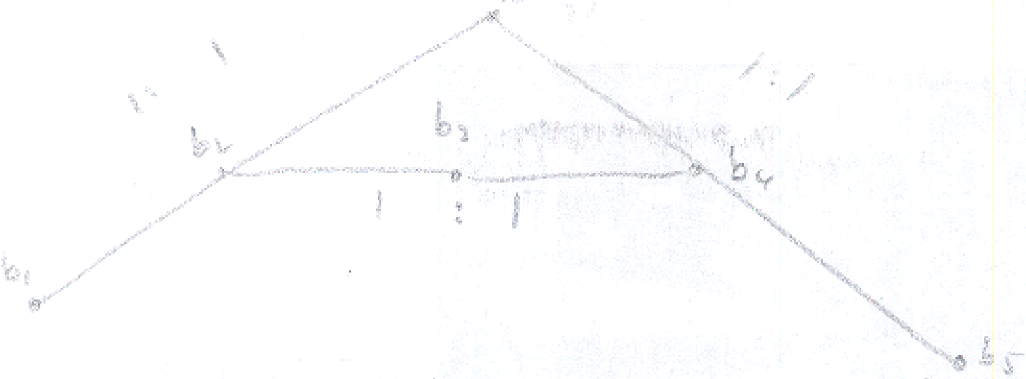
$$b_2^1 = 0.4\,b_2 + 0.6\,b_3 = 0.4\begin{pmatrix}3\\1\end{pmatrix} + 0.6\begin{pmatrix}4\\-1\end{pmatrix} = \begin{pmatrix}1.2\\0.4\end{pmatrix} + \begin{pmatrix}2.4\\-0.6\end{pmatrix} = \begin{pmatrix}3.6\\-0.2\end{pmatrix}$$

$$b_0^2 = 0.4\,b_0^1 + 0.6\,b_1^1 = 0.4\begin{pmatrix}1.6\\1.2\end{pmatrix} + 0.6\begin{pmatrix}2.6\\1.4\end{pmatrix} = \begin{pmatrix}2.2\\1.32\end{pmatrix}$$

$$b_1^2 = 0.4\,b_1^1 + 0.6\,b_2^1 = 0.4\begin{pmatrix}2.6\\1.4\end{pmatrix} + 0.6\begin{pmatrix}3.6\\-0.2\end{pmatrix} = \begin{pmatrix}3.2\\0.44\end{pmatrix}$$

$$b_0^3 = 0.4\,b_0^2 + 0.6\,b_1^2 = 0.4\cdot\begin{pmatrix}2.2\\1.32\end{pmatrix} + 0.6\begin{pmatrix}3.2\\0.44\end{pmatrix} = \begin{pmatrix}2.8\\0.792\end{pmatrix}$$

c) Assuming equidistant parametrization, find the next two control points b_4 and b_5 for a Bézier spline composed of this segment and a subsequent one defined by b_4, b_5, b_6, and b_7 so that the transition between the two segments is C^2.



Using the above cartoon we can determine b_4 directly:

$$b_4 = b_3 + (b_3 - b_2) = 2b_3 - b_2 = 2\cdot\begin{pmatrix}4\\-1\end{pmatrix} - \begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}8-3\\-2-1\end{pmatrix} = \begin{pmatrix}5\\-3\end{pmatrix}$$

Similarly, d can be computed:

$$d = b_2 + (b_2 - b_1) = 2b_2 - b_1 = 2\cdot\begin{pmatrix}3\\1\end{pmatrix} - \begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}6-2\\2-2\end{pmatrix} = \begin{pmatrix}4\\0\end{pmatrix}$$

Then, b_5 can be calculated as:

$$b_5 = b_4 + (b_4 - d) = 2b_4 - d = 2\cdot\begin{pmatrix}5\\-3\end{pmatrix} - \begin{pmatrix}4\\0\end{pmatrix} = \begin{pmatrix}10-4\\-6-0\end{pmatrix} = \begin{pmatrix}6\\-6\end{pmatrix}$$