Example: knot insertion

Let $x(u)$ be a B-spline defined by the control points

$$d_0 = \begin{pmatrix} 0 \\ 2 \end{pmatrix},\ d_1 = \begin{pmatrix} 0.4 \\ 1.6 \end{pmatrix},\ d_2 = \begin{pmatrix} 1.9 \\ 1.6 \end{pmatrix},\ d_3 = \begin{pmatrix} 4.3 \\ 0 \end{pmatrix},\ d_4 = \begin{pmatrix} 4.3 \\ 1 \end{pmatrix}$$

$$d_5 = \begin{pmatrix} 5.1 \\ 1.6 \end{pmatrix},\ d_6 = \begin{pmatrix} 5.7 \\ 1.6 \end{pmatrix},\ d_7 = \begin{pmatrix} 5.7 \\ 2 \end{pmatrix}$$

and the knot vector

$$(0, 0, 0, 0, 0.3, 0.5, 0.8, 0.8, 1, 1, 1, 1)$$

Convert this cubic B-Spline into a Bézier spline:

insert knot u_4 twice

$$\begin{array}{lllll}
b_0 = d_0 & & & & \\
b_1 = d_1 & \searrow & & & \\
d_2 & \longrightarrow & d_2^{(1)} & \searrow & \\
d_3 & \longrightarrow & d_3^{(1)} & \longrightarrow & d_3^{(2)} \\
d_4 & & & &
\end{array}$$

$$b_0 = \begin{pmatrix} 0 \\ 2 \end{pmatrix},\quad b_1 = \begin{pmatrix} 0.4 \\ 1.6 \end{pmatrix},\quad b_2 = d_2^{(1)} = \begin{pmatrix} 1.3 \\ 1.6 \end{pmatrix},\quad b_3 = d_3^{(2)} = \begin{pmatrix} 2.2 \\ 1.24 \end{pmatrix},\quad b_4 = d_3^{(1)} = \begin{pmatrix} 2.8 \\ 1 \end{pmatrix}$$

$$a_i^{(j)} = \frac{t - u_i}{u_{i+m-j} - u_i}$$

$$d_2^{(1)} = a_2^{(1)} d_2 + (1 - a_2^{(1)}) d_1 = 0.6\, d_2 + 0.4\, d_1 = \begin{pmatrix} 1.3 \\ 1.6 \end{pmatrix} \quad ; a_2^{(1)} = \frac{0.3 - u_2}{u_5 - u_2} = \frac{0.3}{0.5} = 0.6$$

$$d_3^{(1)} = a_3^{(1)} d_3 + (1 - a_3^{(1)}) d_2 = \frac{3}{8} d_3 + \frac{5}{8} d_2 = \begin{pmatrix} 2.8 \\ 1 \end{pmatrix} \quad ; a_3^{(1)} = \frac{0.3 - u_3}{u_6 - u_3} = \frac{0.3}{0.8} = \frac{3}{8}$$

$$d_3^{(2)} = a_3^{(2)} d_3^{(1)} + (1 - a_3^{(2)}) d_2^{(1)} = 0.6\, d_3^{(1)} + 0.4\, d_2^{(1)} = \begin{pmatrix} 2.2 \\ 1.24 \end{pmatrix} \quad ; a_3^{(2)} = \frac{0.3 - u_3}{u_5 - u_3} = \frac{0.3}{0.5} = 0.6$$

insert knot u_5 twice.

$$\begin{array}{lllll}
d_2 & \searrow & & & \\
d_3 & \longrightarrow & d_3^{(1)} & \searrow & \\
d_4 & \longrightarrow & d_4^{(1)} & \longrightarrow & d_4^{(2)} \\
d_5 & & & &
\end{array}$$

$$d_3^{(1)} = a_3^{(1)} d_3 + (1 - a_3^{(1)}) d_2 \quad ; \quad a_3^{(1)} = \frac{0.5 - u_3}{u_6 - u_3} = \frac{0.5}{0.8} = \frac{5}{8}$$

$$= \frac{5}{8} d_3 + \frac{3}{8} d_2 = \begin{pmatrix} 3.5 \\ 0.6 \end{pmatrix} = b_5$$

$$d_4^{(1)} = a_4^{(1)} d_4 + (1 - a_4^{(1)}) d_3 \quad ; \quad a_4^{(1)} = \frac{0.5 - u_4}{u_7 - u_4} = \frac{0.2}{0.5} = 0.4$$

$$= 0.4\, d_4 + 0.6\, d_3 = \begin{pmatrix} 4.3 \\ 0.4 \end{pmatrix} = b_7$$

$$d_4^{(2)} = a_4^{(2)} d_4^{(1)} + (1 - a_4^{(2)}) d_3^{(1)} \quad ; \quad a_4^{(2)} = \frac{0.5 - u_4}{u_6 - u_4} = 0.4$$

$$= 0.4\, d_4^{(1)} + 0.6\, d_3^{(1)} = \begin{pmatrix} 3.76 \\ 0.52 \end{pmatrix} = b_6$$

insert knot u_6 once:

d_4

d_5 — $d_5^{(1)}$

$$d_5^{(1)} = a_5^{(1)} d_5 + (1 - a_5^{(1)}) d_4 \quad ; \quad a_5^{(1)} = \frac{0.8 - u_5}{u_8 - u_5} = \frac{0.3}{0.5} = 0.6$$

$$= 0.6\, d_5 + 0.4\, d_4 = \begin{pmatrix} 4.62 \\ 1.36 \end{pmatrix} = b_9$$

$d_0 \leftrightarrow b_0$

$d_1 \leftrightarrow b_1$

d_2 — $d_2^{(1)} \rightarrow b_2$

d_3 — $d_3^{(1)}$ — $d_3^{(2)} \leftrightarrow b_3$

b_4

d_4 — $d_4^{(1)}$ — $d_4^{(2)}$ $\quad b_5$ ← second $d_3^{(1)}$!

b_6

d_5 — $d_5^{(1)}$ $\quad b_7$

b_8

b_9

$d_6 \quad b_{10}$

b_{11}

$d_7 \leftrightarrow b_{12}$

Then, the Bézier form of the given B-spline is represented by the Bézier points:

$$\{b_0, b_1, b_2, b_3, b_3, b_4, b_5, b_6, b_6, b_7, b_8, b_9, b_9, b_{10}, b_{11}, b_{12}\}$$

Note: by using b_3, b_6, and b_9 twice we avoid ~~the~~ to insert u_4 three times (for example), since $d_4^{(3)} = a_4^{(3)} = 0$