1. (5 + 5 points) Find the quadratic polynomial that interpolates the following points

$$p_0 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad p_1 = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad p_2 = \begin{pmatrix} 4 \\ 26 \end{pmatrix}$$

   (a) by solving the system of linear equations.

   (b) by using the Lagrange polynomials.

$p_0 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad p_1 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \quad p_2 = \begin{pmatrix} 4 \\ 26 \end{pmatrix}$

a) quadratic polynomial: $p(x) = ax^2 + bx + c$

linear equation system:

$$p(0) = 2 \Rightarrow c = 2$$
$$p(1) = 5 \Rightarrow a + b + c = 5$$
$$p(4) = 26 \Rightarrow 16a + 4b + c = 26$$

plugging in $c = 1$:

$$\begin{matrix} a + b = 3 \\ 16a + 4b = 24 \end{matrix} \Leftrightarrow \begin{matrix} a + b = 3 \\ 12a = 12 \end{matrix} \Leftrightarrow \begin{matrix} a + b = 3 \\ a = 1 \end{matrix} \Leftrightarrow a = 1;\ b = 2$$

$$\Rightarrow p(x) = x^2 + 2x + 2$$

b) $L_i(x) = \prod_{k=0;\,k\neq i}^{n} \frac{x - x_k}{x_i - x_k}$

First, compute the basis polynomials:

$x_0 = 0, \quad x_1 = 1, \quad x_2 = 4$

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \cdot \frac{x - x_2}{x_0 - x_2} = \frac{(x-1)(x-4)}{(0-1)(0-4)} = \frac{1}{4}(x^2 - 5x + 4)$$

$$L_1(x) = \frac{x - x_0}{x_1 - x_0} \cdot \frac{x - x_2}{x_1 - x_2} = \frac{(x-0)(x-4)}{(1-0)(1-4)} = -\frac{1}{3}(x^2 - 4x)$$

$$L_2(x) = \frac{x - x_0}{x_2 - x_0} \cdot \frac{x - x_1}{x_2 - x_1} = \frac{(x-0)(x-1)}{(4-0)(4-1)} = \frac{1}{12}(x^2 - x)$$

$$\Rightarrow p(x) = 2 \cdot L_0(x) + 5 \cdot L_1(x) + 26\, L_2(x)$$
$$= \frac{1}{2}(x^2 - 5x + 4) - \frac{5}{3}(x^2 - 4x) + \frac{13}{6}(x^2 - x)$$
$$= \frac{3}{6}(x^2 - 5x + 4) - \frac{10}{6}(x^2 - 4x) + \frac{13}{6}(x^2 - x)$$
$$= \frac{6}{6}x^2 + \frac{12}{6}x + \frac{12}{6} = x^2 + 2x + 2$$

2. (1 + 5 points) Suppose a Bezier segment is defined for $0 \le t \le 1$ by the following control points

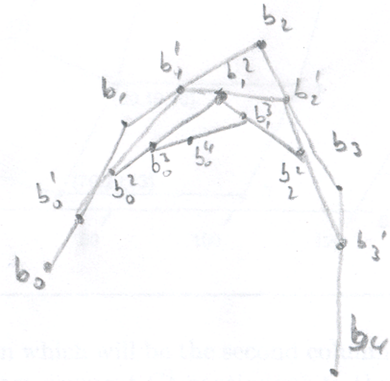
$$b_0 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad b_1 = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad b_2 = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \quad b_3 = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \quad b_4 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

(a) Assuming the Bezier segment is well-defined, of what degree is the resulting polynomial?

(b) Evaluate the Bezier segment at $t = 0.3$ using the de Casteljau algorithm.

a) The resulting Bézier segment is quintic, so the degree $n = 4$.

b)



$b_0^1 = 0.7 b_0 + 0.3 b_1 = \begin{pmatrix} 0.7 \\ 2.9 \end{pmatrix}$

$b_1^1 = 0.7 b_1 + 0.3 b_2 = \begin{pmatrix} 1.9 \\ 5.3 \end{pmatrix}$

$b_2^1 = 0.7 b_2 + 0.3 b_3 = \begin{pmatrix} 4.3 \\ 5.1 \end{pmatrix}$

$b_3^1 = 0.7 b_3 + 0.3 b_4 = \begin{pmatrix} 5 \\ 2.4 \end{pmatrix}$

$b_0^2 = 0.7 b_0^1 + 0.3 b_1^1 = \begin{pmatrix} 0.78 \\ 3.62 \end{pmatrix}$

$b_1^2 = 0.7 b_1^1 + 0.3 b_2^1 = \begin{pmatrix} 2.62 \\ 5.24 \end{pmatrix}$

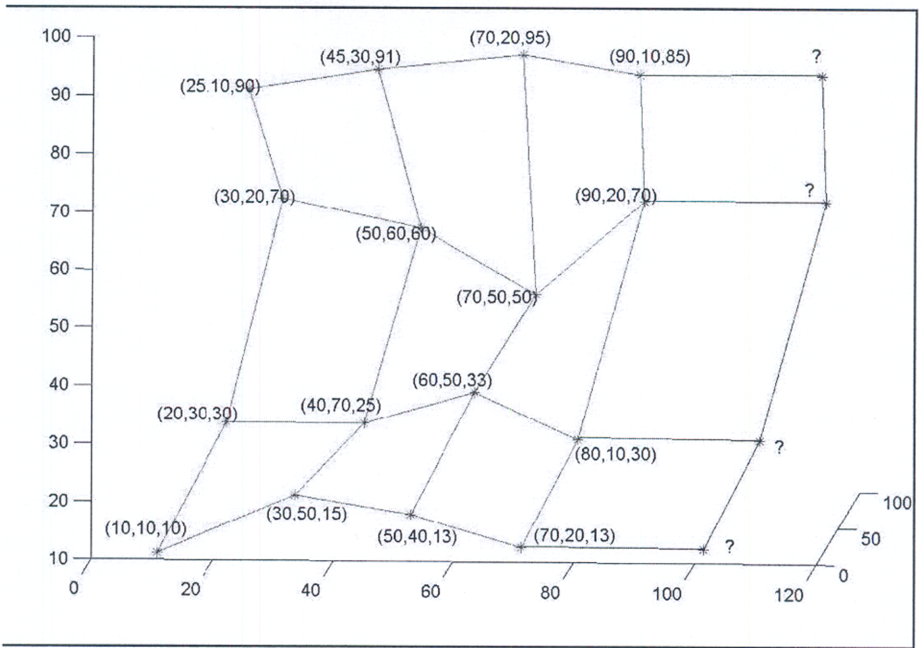
$b_2^2 = 0.7 b_2^1 + 0.3 b_3^1 = \begin{pmatrix} 4.51 \\ 4.29 \end{pmatrix}$

$b_0^3 = 0.7 b_0^2 + 0.3 b_1^2 = \begin{pmatrix} 1.332 \\ 4.106 \end{pmatrix}$

$b_1^3 = 0.7 b_1^2 + 0.3 b_2^2 = \begin{pmatrix} 3.187 \\ 4.555 \end{pmatrix}$

$b_0^4 = 0.7 b_0^3 + 0.3 b_1^3 = \begin{pmatrix} 1.8885 \\ 4.3607 \end{pmatrix}$

3. (5 points) Suppose you have a cubic Bezier surface defined by the following control points.



Find the control points in the last column which will be the second column of control points on another Bezier surface segment C^1 continuous to the original one (assume equidistant parametrization).

For C^1 continuity, each control point has to be along the line joining the last two control points of the existing surface patch. Due to the equidistant parametrization, the distance to the new control points is the same as the distance between the last control points.

Hence, the control points from top down are:

$$\begin{pmatrix} 110 \\ 0 \\ 75 \end{pmatrix} \quad \begin{pmatrix} 110 \\ -10 \\ 90 \end{pmatrix} \quad \begin{pmatrix} 100 \\ -30 \\ 27 \end{pmatrix} \quad \begin{pmatrix} 90 \\ 0 \\ 13 \end{pmatrix}$$

4. (6 points) Prove the equivalence of the quadratic Bernstein polynomials and the quadratic B-spline representation for one segment with multiplicity 3, i.e. show that the quadratic B-spline basis functions for a node vector $(0, 0, 0, 1, 1, 1)$ are identical to the Bernstein basis functions. Per definition, fractions with a denominator of zero are assumed to be zero as well.

$(0, 0, 0, 1, 1, 1)$

$t_0 \; t_1 \; t_2 \; t_3 \; t_4 \; t_5$

Bernstein polynomials

$\binom{2}{i} t^i (1-t)^{2-i}, \quad i = 0, 1, 2$

$(1-t)^2 = 1 - 2t + t^2$

$2t(1-t) = 2t - 2t^2$

t^2

$N_{0,1} = \begin{cases} 1 & t = 0 \\ 0 & \text{else} \end{cases}$

$N_{1,1} = N_{0,1}$

$N_{2,1} = 1$

$N_{3,1} = \begin{cases} 1 & t = 1 \\ 0 & \text{else} \end{cases}$

$N_{4,1} = N_{3,1}$

$N_{0,2} = \frac{t - t_0}{t_1 - t_0} N_{0,1}(t) + \frac{t_2 - t}{t_2 - t_1} N_{1,1}(t) = 0$

$N_{1,2} = \frac{t - t_1}{t_2 - t_1} N_{1,1}(t) + \frac{t_3 - t}{t_3 - t_2} N_{2,1}(t)$

$= (1-t) N_{2,1}(t) = (1-t)$

$N_{2,2} = \frac{t - t_2}{t_3 - t_2} N_{2,1}(t) + \frac{t_4 - t}{t_4 - t_3} N_{3,1}(t)$

$= t \cdot N_{2,1}(t) = t$

$N_{3,2} = \frac{t - t_3}{t_4 - t_3} N_{3,1} + \frac{t_5 - t}{t_5 - t_4} N_{4,1} = 0$

$N_{0,3} = \frac{t - t_0}{t_2 - t_0} N_{0,2} + \frac{t_3 - t}{t_3 - t_1} N_{1,2} = (1-t) N_{1,2} = (1-t)(1-t)$

$N_{1,3} = \frac{t - t_1}{t_3 - t_1} N_{1,2} + \frac{t_4 - t}{t_4 - t_2} N_{2,2} = t N_{1,2} + (1-t) N_{2,2} = t(1-t) + (1-t)t$

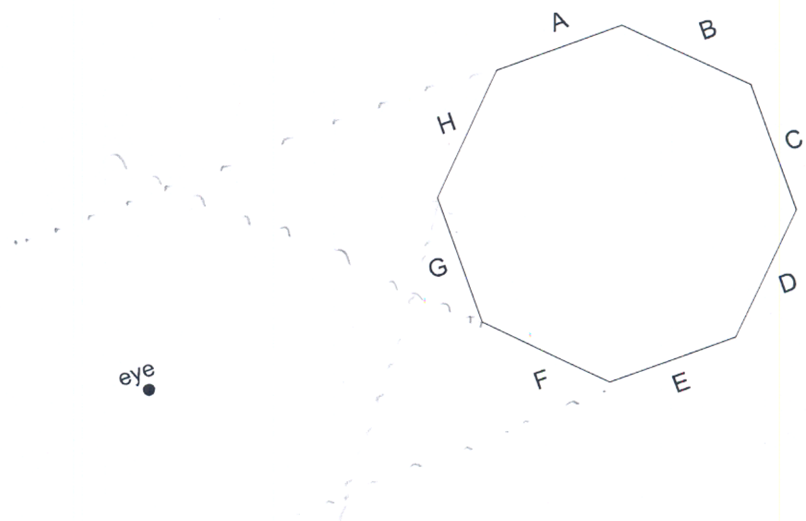
$= 2t(1-t)$

$N_{2,3} = \frac{t - t_2}{t_4 - t_2} N_{2,2} + \frac{t_5 - t}{t_5 - t_3} N_{3,2} = t N_{2,2} = t^2$

5. (2 points) What are the main differences between octress and BSP trees?

Octree	BSP tree
fixed intersection	arbitrary intersection
eight children	two children

6. (2 points) For the following scene, the polygon forming a closed solid object are represented byedges. Wich faces would be removed by back-face culling?



A, B, C, D, E

7. (2 points) What are the components that describe lighting in the Phong illumination model and what do they simulate?

specular: mirror-like reflection

diffuse: reflection in all possible directions

ambient: background light

8. (2 points) In what way do the front and back clipping plane influence the z-buffer algorithm and what error might result from a bad choice for these values?

The front and back clipping plane determine the precision of the z-buffer. Hence, a bad choice can render to polygons indistinguishable with respect to their depth value which might result in flickering.